



AMAT 314
Numerical Methods

Instructor: Marios M. Fyrillas
Email: m.fyrillas@fit.ac.cy
Office Tel.: 22 345159/60 Ext. 131

HOMWORK ASSIGNMENT ON INTERPOLATION

QUESTION 1

Using interpolation by a collocation-polynomial fit method find a second-degree polynomial that fits the data below:

x	f(x)
3.40	0.294118
3.50	0.285714
3.60	0.277778

Verify your calculation using a second-order Lagrange interpolation polynomial.

General Equation

$$y = \alpha_0 + \alpha_1 x + \alpha_2 x^2$$

Step 1

$$0.294118 = \alpha_0 + 3.4\alpha_1 + (3.4)^2\alpha_2 \quad \text{(Equation 1)}$$

$$0.285714 = \alpha_0 + 3.5\alpha_1 + (3.5)^2\alpha_2 \quad \text{(Equation 2)}$$

$$0.277778 = \alpha_0 + 3.6\alpha_1 + (3.6)^2\alpha_2 \quad \text{(Equation 3)}$$

Step 2

$$\begin{pmatrix} 1 & 3.4 & 11.56 \\ 1 & 3.5 & 12.25 \\ 1 & 3.6 & 12.96 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 0.294118 \\ 0.285714 \\ 0.277778 \end{pmatrix}$$

$$a_0 = \frac{\begin{vmatrix} 0.294118 & 3.4 & 11.56 \\ 0.285714 & 3.5 & 12.25 \\ 0.277778 & 3.6 & 12.96 \end{vmatrix}}{\begin{vmatrix} 1 & 3.4 & 11.56 \\ 1 & 3.5 & 12.25 \\ 1 & 3.6 & 12.96 \end{vmatrix}}$$

$$a_0 = \frac{0.294118[(3.5 * 12.96) - (3.6 * 12.25)] - 0.285714[(3.4 * 12.96) - (3.6 * 11.56)] + 0.277778[(3.4 * 12.25) - (3.5 * 11.56)]}{1[(3.5 * 12.96) - (3.6 * 12.25)] - 1[(3.4 * 12.96) - (3.6 * 11.56)] + 1[(3.4 * 12.25) - (3.5 * 11.56)]}$$

$$a_0 = \mathbf{0.8583}$$

Step 3

$$\begin{pmatrix} 1 & 3.4 & 11.56 \\ 1 & 3.5 & 12.25 \\ 1 & 3.6 & 12.96 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 0.294118 \\ 0.285714 \\ 0.277778 \end{pmatrix}$$

$$a_1 = \frac{\begin{vmatrix} 1 & 0.294118 & 11.56 \\ 1 & 0.285714 & 12.25 \\ 1 & 0.277778 & 12.96 \end{vmatrix}}{\begin{vmatrix} 1 & 3.4 & 11.56 \\ 1 & 3.5 & 12.25 \\ 1 & 3.6 & 12.96 \end{vmatrix}}$$

$$a_1 = \frac{1[(0.285714 * 12.96) - (0.277778 * 12.25)] - 1[(0.294118 * 12.96) - (0.277778 * 11.56)] + 1[(0.294118 * 12.25) - (0.285714 * 11.56)]}{1[(3.5 * 12.96) - (3.6 * 12.25)] - 1[(3.4 * 12.96) - (3.6 * 11.56)] + 1[(3.4 * 12.25) - (3.5 * 11.56)]}$$

$$a_1 = -0.24$$

Step 4

$$0.294118 = \alpha_0 + 3.4\alpha_1 + (3.4)^2\alpha_2 \quad \text{(Equation 1)}$$

$$\alpha_0 = 0.85$$

$$\alpha_1 = -0.24$$

$$\Rightarrow (3.4)^2\alpha_2 = 0.294118 - \alpha_0 - 3.4\alpha_1$$

$$\Rightarrow 11.56\alpha_2 = 0.294118 - 0.85 - (3.4)(-0.24)$$

$$\Rightarrow \alpha_2 = 0.02$$

Verify the calculation using a second-order **Lagrange** interpolation polynomial.

Lagrange Equation

$$\begin{aligned} P_{2(x)} &= \frac{(x-x_2)(x-x_3)}{(x_1-x_2)(x_1-x_3)} * y_1 + \frac{(x-x_1)(x-x_3)}{(x_2-x_1)(x_2-x_3)} * y_2 + \frac{(x-x_1)(x-x_2)}{(x_3-x_1)(x_3-x_2)} * y_3 \\ &= \frac{(x-3.5)(x-3.6)}{(3.4-3.5)(3.4-3.6)} * 0.294118 + \frac{(x-3.4)(x-3.6)}{(3.5-3.4)(3.5-3.6)} * 0.285714 + \frac{(x-3.4)(x-3.5)}{(3.6-3.4)(3.6-3.5)} * 0.277778 \\ &= \frac{(x^2 - 3.5x - 3.6x + 12.6)}{0.02} * 0.294118 + \frac{(x^2 - 3.4x - 3.6x + 12.24)}{-0.01} * 0.285714 + \frac{(x^2 - 3.4x - 3.5x + 11.9)}{0.02} * 0.277778 \\ &= (x^2 - 3.5x - 3.6x + 12.6) * 14.7059 + (x^2 - 3.4x - 3.6x + 12.24) * (-28.5714) + (x^2 - 3.4x - 3.5x + 11.9) * 13.8889 \\ &= 14.7059x^2 - 104.41189x + 185.29434 - 28.5714x^2 + 199.9998x - 349.713936 + 13.8889x^2 - 95.83341x + 165.27791 \\ &= \mathbf{0.02x^2 - 0.24x + 0.8583} \end{aligned}$$

This result agrees with the previous results obtained through collocation method.

QUESTION 2

The data in the table seem to fit a cubic equation. Determine by least squares the optimum degree of polynomial.

x	$f(x)$	x	$f(x)$	x	$f(x)$
0.1	1.9	6.6	36.1	13.2	-48.6
1.1	7.9	7.1	23.7	14.1	-40.2
1.6	24.9	8.2	13.0	15.6	-51.6
2.4	24.9	9.1	20.5	16.1	-30.5
2.5	34.9	9.4	-3.1	17.6	-34.6
4.1	42.7	11.1	-13.0	17.9	-16.4
5.2	29.7	11.4	-28.7	19.1	-13.8
6.1	49.8	12.2	-39.5	20.0	-1.1

The equation has the form

$$y = a_0 + a_1x + a_2x^2 + a_3x^3.$$

The objective is to find the optimum coefficients such that this represents a least square fit.

For a general fit of the form

$$y = a_0 + a_1x + a_2x^2 + \dots + a_nx^n$$

the Normal Equations are as follows

$$\begin{array}{ccccccc} a_0N & +a_1 \sum x_i & +a_2 \sum x_i^2 & + \dots & + & a_n \sum x_i^n & = \sum y_i \\ a_0 \sum x_i & +a_1 \sum x_i^2 & +a_2 \sum x_i^3 & + \dots & + & a_n \sum x_i^{n+1} & = \sum x_i y_i \\ a_0 \sum x_i^2 & +a_1 \sum x_i^3 & +a_2 \sum x_i^4 & + \dots & + & a_n \sum x_i^{n+2} & = \sum x_i^2 y_i \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_0 \sum x_i^n & +a_1 \sum x_i^{n+1} & +a_2 \sum x_i^{n+2} & + \dots & + & a_n \sum x_i^{2n} & = \sum x_i^n y_i \end{array}$$

where N is the number of points and $n = N - 1$.

In this case we only need the first four equations

$$\begin{array}{cccccc} a_0N & +a_1 \sum x_i & +a_2 \sum x_i^2 & +a_3 \sum x_i^3 & = & \sum y_i \\ a_0 \sum x_i & +a_1 \sum x_i^2 & +a_2 \sum x_i^3 & +a_3 \sum x_i^4 & = & \sum x_i y_i \\ a_0 \sum x_i^2 & +a_1 \sum x_i^3 & +a_2 \sum x_i^4 & +a_3 \sum x_i^5 & = & \sum x_i^2 y_i \\ a_0 \sum x_i^3 & +a_1 \sum x_i^4 & +a_2 \sum x_i^5 & +a_3 \sum x_i^6 & = & \sum x_i^3 y_i \end{array}$$

Based on the results of the table on the next page the system of equation is as follows:

$$\begin{array}{cccccc} a_0 24 & +a_1 231.8 & +a_2 3101.84 & +a_3 47145.212 & = & -11.1 \\ a_0 231.8 & +a_1 3101.84 & +a_2 47145.212 & +a_3 768326.1452 & = & -3146.27 \\ a_0 3101.84 & +a_1 47145.212 & +a_2 768326.1452 & +a_3 13080405.13 & = & -60707.221 \\ a_0 47145.212 & +a_1 768326.1452 & +a_2 13080405.13 & +a_3 229355836.8 & = & -1008994.255 \end{array}$$

The resulting equation is as follows $f = -6.355 + 24.086x - 3.611x^2 + 0.122x^3$

	x	y	x^2	x^3	x^4	x^5	x^6	xy	x^2y	x^3y
	0.1	1.9	0.01	0.001	0.0001	0.00001	0.000001	0.19	0.019	0.0019
	1.1	7.9	1.21	1.331	1.4641	1.61051	1.771561	8.69	9.559	10.5149
	1.6	24.9	2.56	4.096	6.5536	10.48576	16.777216	39.84	63.744	101.9904
	2.4	24.9	5.76	13.824	33.1776	79.62624	191.102976	59.76	143.424	344.2176
	2.5	34.9	6.25	15.625	39.0625	97.65625	244.140625	87.25	218.125	545.3125
	4.1	42.7	16.81	68.921	282.5761	1158.56201	4750.104241	175.07	717.787	2942.9267
	5.2	29.7	27.04	140.608	731.1616	3802.04032	19770.60966	154.44	803.088	4176.0576
	6.1	49.8	37.21	226.981	1384.5841	8445.96301	51520.37436	303.78	1853.058	11303.6538
	6.6	36.1	43.56	287.496	1897.4736	12523.32576	82653.95002	238.26	1572.516	10378.6056
	7.1	23.7	50.41	357.911	2541.1681	18042.29351	128100.2839	168.27	1194.717	8482.4907
	8.2	13	67.24	551.368	4521.2176	37073.98432	304006.6714	106.6	874.12	7167.784
	9.1	20.5	82.81	753.571	6857.4961	62403.21451	567869.252	186.55	1697.605	15448.2055
	9.4	-3.1	88.36	830.584	7807.4896	73390.40224	689869.7811	-29.14	-273.916	-2574.8104
	11.1	-13	123.21	1367.631	15180.7041	168505.8155	1870414.552	-144.3	-1601.73	-17779.203
	11.4	28.7	129.96	1481.544	16889.6016	192541.4582	2194972.624	-327.18	-3729.852	-42520.3128
	12.2	39.5	148.84	1815.848	22153.3456	270270.8163	3297303.959	-481.9	-5879.18	-71725.996
	13.2	48.6	174.24	2299.968	30359.5776	400746.4243	5289852.801	-641.52	-8468.064	111778.4448
	14.1	40.2	198.81	2803.221	39525.4161	557308.367	7858047.975	-566.82	-7992.162	112689.4842
	15.6	51.6	243.36	3796.416	59224.0896	923895.7978	14412774.45	-804.96	12557.376	195895.0656
	16.1	30.5	259.21	4173.281	67189.8241	1081756.168	17416274.3	-491.05	-7905.905	127285.0705
	17.6	34.6	309.76	5451.776	95951.2576	1688742.134	29721861.55	-608.96	10717.696	188631.4496
	17.9	16.4	320.41	5735.339	102662.5681	1837659.969	32894113.44	-293.56	-5254.724	-94059.5596
	19.1	13.8	364.81	6967.871	133086.3361	2541949.02	48551226.27	-263.58	-5034.378	-96156.6198
	20	-1.1	400	8000	160000	3200000	64000000	-22	-440	-8800
SUM	231.8	11.1	3101.84	47145.212	768326.1452	13080405.13	229355836.8	3146.27	60707.221	1008994.255

QUESTION 3

Half-wave rectifier:

A sinusoidal voltage $E \sin(\omega t)$, where t is time, is passed through a half-wave rectifier that clips the negative portion of the wave. The resulting function is

$$u = \begin{cases} 0 & \text{if } -L/2 < t < 0 \\ E \sin(\omega t) & \text{if } 0 < t < L/2 \end{cases}$$

- Determine L for the case $E = 1$ and $\omega = 2\pi$ and plot the function.**
- Find the Fourier series of the resulting periodic function by direct integration.**
- Find the finite Fourier series by discretizing the function in 8, 16 and 32 points.**

$$F(t) = \frac{a_0}{2} + \sum_k \left[a_k \cos\left(\frac{2\pi}{L} kt\right) + b_k \sin\left(\frac{2\pi}{L} kt\right) \right]$$

where L is the period of the signal

and the coefficients a_k and b_k are given by

$$a_k = \frac{2}{L} \int_0^L F(t) \cos\left(\frac{2\pi}{L} kt\right) dt$$

$$b_k = \frac{2}{L} \int_0^L F(t) \sin\left(\frac{2\pi}{L} kt\right) dt$$

$$\frac{2\pi}{L} = \omega = 2\pi \Rightarrow L = 1$$

$$a_k = 2 \int_0^1 f(t) \cos(2\pi kt) dt, \quad b_k = 2 \int_0^1 f(t) \sin(2\pi kt) dt$$

$$\Rightarrow a_0 = 2 \int_0^1 f(t) dt, \quad a_1 = 2 \int_0^1 f(t) \cos(2\pi t) dt, \quad b_1 = 2 \int_0^1 f(t) \sin(2\pi t) dt$$

$$a_0 = 2 \int_0^{1/2} f(t) dt + \int_{1/2}^1 f(t) dt = 2 \int_0^{1/2} \sin(2\pi t) dt = -\frac{2 \cos(2\pi t)}{2\pi} \Big|_0^{1/2} = -\frac{\cos(\pi)}{\pi} + \frac{1}{\pi} = \frac{2}{\pi}$$

$$a_1 = 2 \int_0^{1/2} f(t) \cos(2\pi t) dt + \int_{1/2}^1 f(t) \cos(2\pi t) dt = 2 \int_0^{1/2} \sin(2\pi t) \cos(2\pi t) dt = 0$$

$$b_1 = 2 \int_0^{1/2} f(t) \sin(2\pi t) dt + 2 \int_{1/2}^1 f(t) \sin(2\pi t) dt = 2 \int_0^{1/2} \sin^2(2\pi t) dt = \frac{1}{2}$$

QUESTION 4

The data in the table seem to fit a quadratic equation. Determine by least squares the optimum coefficients if you know that the curve must pass through the point (0,1).

x	$f(x)$	x	$f(x)$	x	$f(x)$
0.05	0.956	0.46	0.571	0.82	0.306
0.11	0.890	0.52	0.539	0.98	0.242
0.15	0.832	0.70	0.378	1.17	0.104
0.31	0.717	0.74	0.370		

If the line must pass form the point (0,1), then it must be of the form $y = 1 + a_1 x + a_2 x^2$, i.e. $a_0 = 1$. The Normal Equations are as follows

$$\begin{aligned} 1 \sum x_i + a_1 \sum x_i^2 + a_2 \sum x_i^3 &= \sum x_i y_i \\ 1 \sum x_i^2 + a_1 \sum x_i^3 + a_2 \sum x_i^4 &= \sum x_i^2 y_i \end{aligned}$$

QUESTION 5

In an experiment the following set of data were collected:

x	$f(x)$	x	$f(x)$	x	$f(x)$
0.1	91	6.6	6550	13.2	13001
1.1	1112	7.1	7155	14.1	14198
1.6	1565	8.2	8263	15.6	15600

- a. The data seem to fit a linear equation. Determine by least squares the coefficients of the equation.

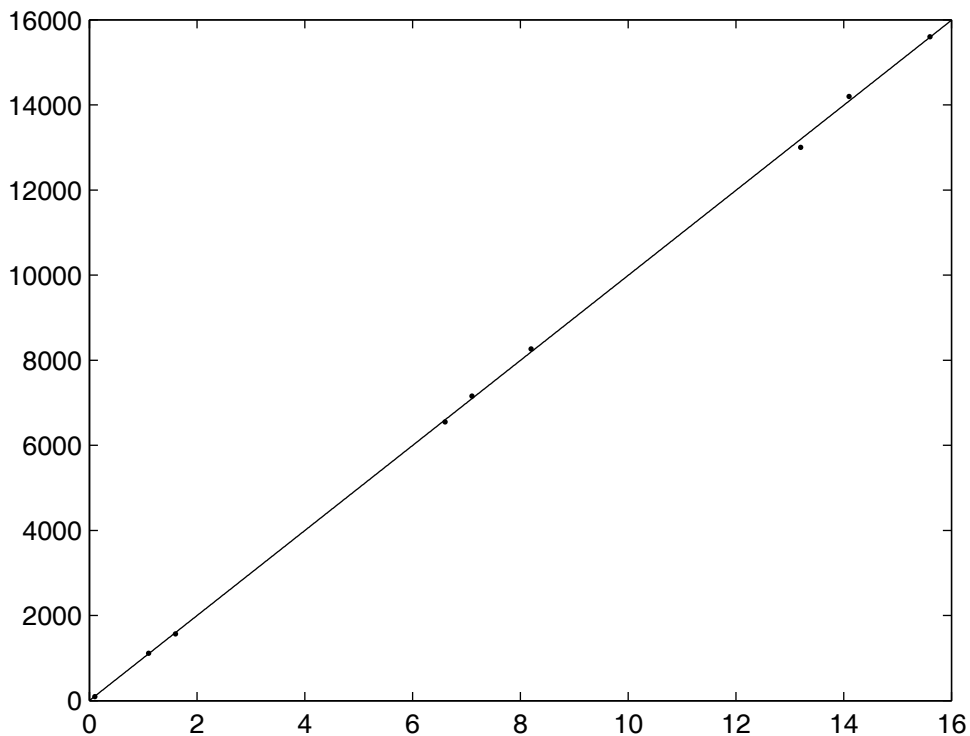
We fit a linear equation of the form $y = a_0 + a_1 x$ hence we need only the terms that involve a_0 and a_1 :

$$\begin{aligned} a_0 N + a_1 \sum x_i &= \sum y_i \\ a_0 \sum x_i + a_1 \sum x_i^2 &= \sum x_i y_i \end{aligned}$$

Substitute the results:

$$\begin{aligned} 9 a_0 + 67.6 a_1 &= 67535 \\ 67.6 a_0 + 781.4 a_1 &= 780688 \end{aligned}$$

Solve the above system to find $a_0 = 1.08$ and $a_1 = 999.2$



- b. **Determine the linear equation if you know that the line must pass through the point (0,0).**

If the line must pass form the point (0,0), then it must be of the form $y = a_1 x$, i.e. $a_0 = 0$. To determine the appropriate form of the equations one must go back to theory. The error between the line and each point is

$$e_1 = y_1 - a_1 x_1$$

$$e_2 = y_2 - a_1 x_2$$

$$e_3 = y_3 - a_1 x_3$$

etc

$$e_9 = y_9 - a_1 x_9$$

To find the optimum line, I have to minimize the sum of the squares of the error, i.e. minimize $S = (e_1^2 + e_2^2 + e_3^2 + \dots + e_9^2)$. At the minimum the derivative must be zero

$$\frac{dS}{d a_1} = 0 = 2(y_1 - a_1 x_1)(-x_1) + 2(y_2 - a_1 x_2)(-x_2) + \dots$$

$$\Rightarrow (-2)(y_1 - a_1 x_1)(x_1) + (y_2 - a_1 x_2)(x_2) + \dots = 0$$

$$\Rightarrow (y_1 - a_1 x_1)x_1 + (y_2 - a_1 x_2)x_2 + \dots = 0$$

$$\Rightarrow y_1 x_1 - a_1 x_1^2 + y_2 x_2 - a_1 x_2^2 + \dots = 0$$

$$\Rightarrow a_1 x_1^2 + a_1 x_2^2 + \dots = y_1 x_1 + y_2 x_2 + \dots$$

$$\Rightarrow a_1 = \frac{\sum x_i y_i}{\sum x_i^2} = \frac{780688}{781.4} = 999.1$$

QUESTION 6

(a) Using interpolation by a Direct Fit Polynomial (collocation polynomial method) find a third-degree polynomial that fits the data below:

x	f(x)
1	0
-2	15
-1	0
2	9

(b) Verify your result with a third-order Lagrange polynomial. Multiply out each term to express in standard form, $ax^3 + bx^2 + cx + d$

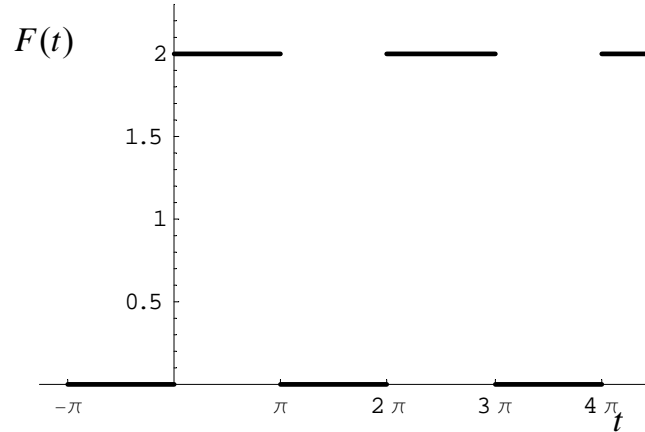
(c) Using least-square regression, determine the optimum quadratic polynomial.

QUESTION 7

Determine the Fourier Series:

$$F(t) = \frac{a_0}{2} + \sum_k \left[a_k \cos\left(\frac{2\pi}{L}kt\right) + b_k \sin\left(\frac{2\pi}{L}kt\right) \right]$$

for the rectangular pulse shown on the figure:



Solution:

$$a_k = \frac{2}{L} \int_0^L F(t) \cos\left(\frac{2\pi}{L}kt\right) dt$$

$$b_k = \frac{2}{L} \int_0^L F(t) \sin\left(\frac{2\pi}{L}kt\right) dt$$

The period L is 2π !

$$a_0 = \frac{2}{L} \int_0^L F(t) dt = \frac{2}{2\pi} \int_0^\pi F(t) dt + \frac{2}{2\pi} \int_\pi^{2\pi} F(t) dt = \frac{1}{\pi} \int_0^\pi 2 dt = \frac{2}{\pi} t \Big|_0^\pi = \frac{2\pi}{\pi} = 2$$

$$\begin{aligned} a_k &= \frac{2}{L} \int_0^L F(t) \cos\left(\frac{2\pi}{L}kt\right) dt = \frac{2}{2\pi} \int_0^\pi 2 \cos(kt) dt + \frac{2}{2\pi} \int_\pi^{2\pi} 0 \cos(kt) dt \\ &= \frac{2}{\pi} \int_0^\pi \cos(kt) dt = \frac{2}{\pi k} \sin(kt) \Big|_0^\pi = \frac{2}{\pi k} [\sin(\pi k) - \sin(0)] = 0 \end{aligned}$$

$$b_k = \frac{2}{L} \int_0^L F(t) \sin\left(\frac{2\pi}{L}kt\right) dt = \frac{2}{2\pi} \int_0^\pi 2 \sin(kt) dt + \frac{2}{2\pi} \int_\pi^{2\pi} 0 \sin(kt) dt$$

$$= \frac{2}{\pi} \int_0^\pi \sin(kt) dt = -\frac{2}{\pi k} \cos(kt) \Big|_0^\pi = \frac{2}{\pi k} [\cos(0) - \cos(\pi k)] = \frac{2}{\pi k} [1 - \cos(\pi k)] = \begin{cases} \text{if } k \text{ even, i.e. } 2,4,6 \dots = 0 \\ \text{if } k \text{ odd, i.e. } 1,3,5 \dots = \frac{4}{\pi k} \end{cases}$$

$$\text{Hence } F(t) = 1 + \frac{4}{\pi} \sum_{k=1,3,5,\dots} \frac{\sin(kt)}{k}$$

QUESTION 8

Given the function $y = f(x)$ the inverse function is defined as $x = f^{-1}(y)$. Show that the inverse function can be approximated to second order through the polynomial:

$$x = \frac{(y - y_1)(y - y_2)}{(y_0 - y_1)(y_0 - y_2)} x_0 + \frac{(y - y_0)(y - y_2)}{(y_1 - y_0)(y_1 - y_2)} x_1 + \frac{(y - y_0)(y - y_1)}{(y_2 - y_0)(y_2 - y_1)} x_2$$

where (x_0, y_0) , (x_1, y_1) and (x_2, y_2) are three selected points. Hence, develop a second order algorithm through which you can obtain the root of the equation $f(x) = 0$.

Above equation is in the form of a second order Lagrange polynomial, hence it passes through the points (x_0, y_0) , (x_1, y_1) and (x_2, y_2) .

As we have inverted x with y , above polynomial represents the inverse function.

The equation $f(x) = 0$ is associated with the point $y = 0$ of the equation $y = f(x)$. Hence, by setting $y = 0$ in the above equation we obtain the point

$$x_3 = \frac{(-y_1)(-y_2)}{(y_0 - y_1)(y_0 - y_2)} x_0 + \frac{(-y_0)(-y_2)}{(y_1 - y_0)(y_1 - y_2)} x_1 + \frac{(-y_0)(-y_1)}{(y_2 - y_0)(y_2 - y_1)} x_2.$$

The algorithm is as follows:

$$x_{n+1} = \frac{y_{n-1}y_n}{(y_{n-2} - y_{n-1})(y_{n-2} - y_n)} x_{n-2} + \frac{y_{n-2}y_n}{(y_{n-1} - y_{n-2})(y_{n-1} - y_n)} x_{n-1} + \frac{y_{n-2}y_{n-1}}{(y_n - y_{n-2})(y_n - y_{n-1})} x_n$$