FREDERICK UNIVERSITY CYPRUS

AMAT 314
Numerical Methods

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HOMEWORK ASSIGNMENT ON NUMERICAL DIFFERENTIATION/ODE

Question 1

Solve the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = x + y + xy, \quad y(0) = 1$$

by Taylor-series expansion to get the value of y **at** $x = 0.1$ **and at** $x = 0.5$**. Use the terms through** x^5**.**

$$y(x) = y(0) + x\,y'(0) + \frac{x^2}{2!}y''(0) + \frac{x^3}{3!}y'''(0) + \frac{x^4}{4!}y''''(0) + \frac{x^5}{5!}y'''''(0)$$

$$y(0) = 1 \text{ (given)}$$

$$y'(x) = \frac{\mathrm{d}y}{\mathrm{d}x} = x + y + xy \;\Rightarrow\; y'(0) = 0 + y(0) + 0\cdot y(0) = 0 + 1 + 0 = 1$$

$$y''(x) = \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 1 + \frac{\mathrm{d}y}{\mathrm{d}x} + x\frac{\mathrm{d}y}{\mathrm{d}x} + y \Rightarrow y''(0) = 1 + y'(0) + x\cdot y'(0) + y(0) = 1 + 1 + 0\cdot 1 + 1 = 3$$

$$y'''(x) = \frac{\mathrm{d}^3y}{\mathrm{d}x^3} = \frac{\mathrm{d}^2y}{\mathrm{d}x^2} + x\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \frac{\mathrm{d}y}{\mathrm{d}x} + \frac{\mathrm{d}y}{\mathrm{d}x} \Rightarrow y'''(0) = y''(0) + x\cdot y''(0) + y'(0) + +y'(0) = 5$$

$$y''''(0) = y'''(0) + x\cdot y'''(0) + 3y''(0) = 14$$

$$y'''''(0) = y''''(0) + x\cdot y''''(0) + 4y'''(0) = 34$$

$$\Rightarrow$$

$$y(x) = 1 + x\,1 + \frac{x^2}{2!}3 + \frac{x^3}{3!}5 + \frac{x^4}{4!}14 + \frac{x^5}{5!}34 = 1 + x + \frac{3}{2}x^2 + \frac{5}{6}x^3 + \frac{7}{12}x^4 + \frac{17}{60}x^5$$

$$y(0.1) = 1.11589, \; y(0.5) = 2.0245$$

Question 2

Use the simple Euler method to solve for $y(0.1)$ **from**

$$\frac{\mathrm{d}y}{\mathrm{d}x} = x + y + xy, \quad y(0) = 1$$

with $\delta x = 0.01$**. Compare your result to the value determined by Taylor series in the previous exercise, estimate how small** δx **would need to be to obtain four-decimal accuracy.**

$y(x_0 + h \cdot n) = y\left(x_0 + h \cdot (n-1)\right) + h \cdot y'\left(x_0 + h \cdot (n-1)\right)$

n	$x + h \cdot (n-1)$	$y\left(x + h \cdot (n-1)\right)$	$y'\left(x + h \cdot (n-1)\right)$	$y\left(x + h \cdot (n-1)\right) +$ $h \cdot y'\left(x + h \cdot (n-1)\right)$
1	0	1	1	1.01
2	0.01	1.01	1.0301	1.0203
3	0.02	1.0203	1.0607	1.0309
4	0.03	1.0309	1.0918	1.0418
5	0.04	1.0418	1.1235	1.0530
6	0.05	1.0530	1.1557	1.0646
7	0.06	1.0646	1.19	1.0765
8	0.07	1.0765	1.22	1.0887
9	0.08	1.0887	1.26	1.1013
10	0.09	1.1013	1.29	1.1142
11	0.1	**1.1142**		

Question 3

Use the simple Euler method to solve

$$y' = \sin x + y, \; y(0) = 2$$

to obtain y **at** $x = 0.5$ **using a step of 0.1**

Question 4

Use finite difference quotients to solve

$$y'' - xy' = e^x \qquad y(0) = 1, \; y(1) = -1$$

Use $h = 0.2$**.**

Finite difference scheme is as follows:

$\dfrac{y_{i+1} - 2y_i + y_{i-1}}{h^2} - x_i \dfrac{y_{i+1} - y_{i-1}}{2h} = e^{x_i}$ which simplifies to

$(2 + h \cdot x_i)y_{i-1} - 4y_i + (2 - h \cdot x_i)y_{i+1} = 2 \cdot h^2 \cdot e^{x_i}$

i	x_i	y_i
0	$x_0 = 0$	$y_0 = 1$
1	$x_1 = 0.2$	y_1
2	$x_2 = 0.4$	y_2
3	$x_3 = 0.6$	y_3
4	$x_4 = 0.8$	y_4
5	$x_5 = 1.0$	$y_5 = -1$

Evaluate the finite difference scheme at each interior point:

$$i=1 \quad (2+0.2\cdot x_1)y_0 - 4y_1 + (2-0.2\cdot x_1)y_2 = 2\cdot 0.2^2 e^{x_1}$$
$$i=2 \quad (2+0.2\cdot x_2)y_1 - 4y_2 + (2-0.2\cdot x_2)y_3 = 2\cdot 0.2^2 e^{x_2}$$
$$i=3 \quad (2+0.2\cdot x_3)y_2 - 4y_3 + (2-0.2\cdot x_3)y_4 = 2\cdot 0.2^2 e^{x_3}$$
$$i=4 \quad (2+0.2\cdot x_4)y_3 - 4y_4 + (2-0.2\cdot x_4)y_5 = 2\cdot 0.2^2 e^{x_4}$$

$$i=1 \quad (2+0.2\cdot 0.2)\cdot(1) - 4y_1 + (2-0.2\cdot 0.2)y_2 = 2\cdot 0.2^2 e^{0.2}$$
$$i=2 \quad (2+0.2\cdot 0.4)y_1 - 4y_2 + (2-0.2\cdot 0.4)y_3 = 2\cdot 0.2^2 e^{0.4}$$
$$i=3 \quad (2+0.2\cdot 0.6)y_2 - 4y_3 + (2-0.2\cdot 0.6)y_4 = 2\cdot 0.2^2 e^{0.6}$$
$$i=4 \quad (2+0.2\cdot 0.8)y_3 - 4y_4 + (2-0.2\cdot 0.8)\cdot(-1) = 2\cdot 0.2^2 e^{0.8}$$

$$i=1 \quad -4y_1 + 1.96y_2 = -1.9423$$
$$i=2 \quad 2.08y_1 - 4y_2 + 1.92y_3 = 0.1193$$
$$i=3 \quad 2.12y_2 - 4y_3 + 1.88y_4 = 0.1458$$
$$i=4 \quad 2.16y_3 - 4y_4 = 2.018$$

$$\begin{pmatrix} -4 & 1.96 & 0 & 0 \\ 2.08 & -4 & 1.92 & 0 \\ 0 & 2.12 & -4 & 1.88 \\ 0 & 0 & 2.16 & -4 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} -1.9423 \\ 0.1193 \\ 0.1458 \\ 2.018 \end{pmatrix}$$

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} 0.5422 \\ 0.1155 \\ -0.2846 \\ -0.6582 \end{pmatrix}$$

Question 5

Use central-difference approximations to solve the boundary value problem:

$$\frac{d^2x}{dt^2} - \left(1 - \frac{t}{5}\right)x = t, \quad x(1) = 2, \quad x(3) = -1.$$

Use a step-size of 0.2