AMAT 314
Numerical Methods

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HOMEWORK ASSIGNMENT ON NUMERICAL INTEGRATION/DIFFERENTIATION

QUESTION 1

The following values of a function are given:

x	$f(x)$	x	$f(x)$
1.0	**1.543**	**1.5**	**2.352**
1.1	**1.668**	**1.6**	**2.577**
1.2	**1.811**	**1.7**	**2.828**
1.3	**1.971**	**1.8**	**3.107**
1.4	**2.151**		

Integrate numerically the function tabulated in the Table over the interval $x = 1.0$ **to** $x = 1.8$**, i.e.** $\int_{1.0}^{1.8} f(x)\,\mathrm{d}x$**, using the trapezoidal rule with an interval** $h = 0.1$ **and** $h = 0.2$**. Use Romberg integration to improve the result.**

The trapezoidal rule is as follows:

$$\mathrm{NI}(\delta x = 0.2) = \frac{\delta x}{2}\left(f(1) + 2f(1.2) + \ldots + 2f(1.6) + f(1.8)\right) =$$

$$\mathrm{NI}(\delta x = 0.1) = \frac{\delta x}{2}\left(f(1) + 2f(1.1) + 2f(1.2) + \ldots + 2f(1.7) + f(1.8)\right) =$$

Use Romberg Extrapolation to improve the result:

If we assume that the error is of the form $c*(\delta x)^2$ then we have the following system of equations:

$$\mathrm{I} = \mathrm{NI}(\delta x = 0.2) + c*0.2^2$$

$$\mathrm{I} = \mathrm{NI}(\delta x = 0.1) + c*0.1^2$$

Solve the system to find that $\mathbf{I = 0.946083}$.

QUESTION 2

Integrate numerically the function tabulated in the Table over the interval $x = 1.8$ to $x = 3.4$, i.e.

$$I = \int_{1.8}^{3.4} f(x)\,\mathrm{d}x .$$

x	$y,\ f(x)$	x	$y,\ f(x)$
1.6	**4.953**	**2.8**	**16.445**
1.8	**6.050**	**3.0**	**20.086**
2.0	**7.389**	**3.2**	**24.533**
2.2	**9.025**	**3.4**	**29.964**
2.4	**11.023**	**3.6**	**36.598**
2.6	**13.464**	**3.8**	**44.701**

Use trapezoidal rule with an interval $h = 0.2$ and $h = 0.4$. Use Romberg integration to improve the result.

QUESTION 3

The second-degree Newton-Cotes formula integrates a quadratic over 2 intervals that are of uniform width h. (Such intervals are often called *panels*.) The equation is

$$\int_{x_0}^{x_2} f(x)\,\mathrm{d}x \approx \frac{h}{3}\left(f_0 + 4 f_1 + f_2 \right) .$$

Use this formula to evaluate the integral in Question 1.

QUESTION 4

Use the Lagrange Polynomial of order 2, i.e.

$$P_2(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} f_0 + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} f_1 + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} f_2$$

to derive the second-degree Newton-Cotes formula mentioned in Question 2.

QUESTION 5

Compute

$$\int_0^1 \frac{\sin(x)}{x}\mathrm{d}x$$

using the trapezoidal rule with $\delta x = 0.5$ and with $\delta x = 0.25$.

Note that $\lim_{x\to 0}\frac{\sin(x)}{x} = 1$. (This can be proved by L'Hopital's rule).

$$f(x) = \frac{\sin(x)}{x}$$

The trapezoidal rule is as follows:

$$\mathrm{NI}(\delta x = 0.5) = \frac{\delta x}{2}\left(f(0) + 2f(0.5) + f(1)\right) = 0.25\left(1 + 2*\frac{\sin(0.5)}{0.5} + \frac{\sin(1)}{1}\right) =$$

$$= 0.25*(1 + 2*0.95885 + 0.84147) = \mathbf{0.939793}$$

$$\mathrm{NI}(\delta x = 0.25) = \frac{\delta x}{2}\left(f(0) + 2f(0.25) + 2f(0.5) + 2f(0.75) + f(1)\right) =$$

$$0.25\left(1 + 2*\frac{\sin(0.25)}{0.25} + 2*0.95885 + 2*\frac{\sin(0.75)}{0.75} + 0.84147\right) =$$

$$= 0.25*(1 + 2*0.98961 + 2*0.95885 + 2*0.90885 + 0.84147) = \mathbf{0.94451}$$

Use Romberg Extrapolation to improve the result:

If we define the exact result as $\mathrm{I} = \int_0^1 \frac{\sin(x)}{x}\mathrm{d}x$ and assume that the error is of the form $c*(\delta x)^2$ then we have the following system of equations:

$$\mathrm{I} = \mathrm{NI}(\delta x = 0.25) + c*0.25^2$$

$$\mathrm{I} = \mathrm{NI}(\delta x = 0.5) + c*0.5^2$$

Substitute the results to obtain:

$$\mathrm{I} = 0.94451 + c*0.0625$$

$$\mathrm{I} = 0.93979 + c*0.25$$

Solve the system to find that $\mathbf{I = 0.946083}$.

The exact value of the integral is 0.946083. So, using Romberg integration I was able to use two solutions that were not very accurate to obtain an accurate solution.

<u>QUESTION 6</u>

Compute

$$\int_0^1 \frac{\sin(x)}{x}\mathrm{d}x$$

using Simpson's rule with $\delta x = 0.5$ **and with** $\delta x = 0.25$**.**

h=0.5

$$\int_0^1 \frac{\sin(x)}{x} dx = I$$

$$\left.\begin{aligned}
&I= \frac{h}{3}\left(f(x=0) + 4f(x=0.5) + f(1)\right)\\
&f(x=0) = \lim_{x->0} \frac{\sin(x)}{x} = \lim_{x->0} \cos(x) = 1 \quad \text{(L' Hopitals rule)}\\
&f(x=0.5) = \frac{\sin(0.5)}{0.5} = 0.9589 \text{ (careful! sin(x) expressed in radians)}\\
&f(1) = \frac{\sin(1)}{1} = 0.8415
\end{aligned}\right\} \Rightarrow$$

$$I= \frac{0.5}{3}\left(1 + 4\cdot 0.9589 + 0.8415\right) = 0.946151$$

h=0.25

$$\left.\begin{aligned}
&I= \frac{h}{3}\left(f(x=0) + 4f(x=0.25) + 2f(0.5) + 4f(x=0.75) + f(1)\right)\\
&f(x=0) = 1\\
&f(x=0.25) = 0.9896\\
&f(x=0.5) = 0.9589\\
&f(x=0.75) = 0.9089\\
&f(x=1) = 0.8415
\end{aligned}\right\} \Rightarrow$$

$$I= \frac{0.25}{3}\left(1 + 4\cdot 0.9896 + 2\cdot 0.9589 + 4\cdot 0.9089 + 0.8415\right) = 0.946087$$

QUESTION 7

Consider the definite, improper integral

$$I = \int_0^1 \sqrt{\frac{(x^2+1)}{x}}\, dx \, .$$

If we try to evaluate this integral using Trapezoidal or Simpson's rule we will run into trouble. Why?

Transform the integral using $y = \sqrt{x}$ and apply Simpson's rule to obtain the result. Choose an h small enough to guarantee 3-decimal places accuracy.

$y=\sqrt{x} \Rightarrow \mathrm{d}y=\dfrac{1}{2\sqrt{x}}\mathrm{d}x$. Substitute the transformation in the integral to obtain:

$$\int_0^1 \sqrt{\frac{(x^2+1)}{x}}\,2\sqrt{x}\ \mathrm{d}y = 2\int_0^1 \sqrt{x^2+1}\ \mathrm{d}y = 2\int_0^1 \underbrace{\sqrt{y^4+1}}_{f}\ \mathrm{d}y$$

Simpson rule is fourth order accurate. Hence, to obtain an accuracy of less than 0.001 use a step size of 0.1:

$$I = 2\frac{0.1}{3}\left[\, f(y=0)+4f(y=0.1)+2f(y=0.2)+4f(y=0.3)+2f(y=0.4)+4f(y=0.5)+ \right.$$
$$\left. 2f(y=0.6)+4f(y=0.7)+2f(y=0.8)+4f(y=0.9)+f(y=1.0)\right]$$

$$I=\frac{0.2}{3}\left[1+4\times1.0000+2\times1.0008+4\times1.004+2\times1.0127+4\times1.0308+ \right.$$
$$\left. 2\times1.0628+4\times1.1136+2\times1.1873+4\times1.2869+1.4142\right]=2.1788$$

QUESTION 8

Evaluate $I = \sqrt{2}\int_0^1 \sqrt{(1-\theta)}\ \mathrm{d}\theta$ **using the three-term Gaussian quadrature (table below), and Simpson's rule.**

Number of terms	Point of evaluation	Weighting factor	Valid up to degree
2	-0.57735027	1.0	3
	0.57735027	1.0	
3	-0.77459667	0.5555555	5
	0	0.8888889	
	0.77459667	0.5555555	

Table: Values of the Gaussian Quadrature for $\int_{-1}^{1} f[x]\ \mathrm{d}x$

Using Simpson's Rule with 5 points :

$$I = \frac{\delta\theta}{3}\left(f[\theta_0] + 4f[\theta_1] + 2f[\theta_2] + 4f[\theta_3] + f[\theta_4]\right)$$

Hence, the step must be $\delta\theta = 0.25$.

$$\Rightarrow I = \frac{0.25}{3}\left(f[0] + 4f[0.25] + 2f[0.5] + 4f[0.75] + f[1]\right)$$

$$= 0.083333 \times \sqrt{2}\ \left(\sqrt{1-0} + 4\sqrt{1-0.25} + 2\sqrt{1-0.5} + 4\sqrt{1-0.75} + \sqrt{0}\right)$$

$$= 0.11785\ (1 + 3.464 + 1.4142 + 2 + 0) = \boxed{0.92845}$$

Using 3 - point Gaussian Quadrature:

Change the variable such that the limits of the integration are -1 and 1,

i.e. x=$2\theta - 1 \Rightarrow \mathrm{d}x = 2\ \mathrm{d}\theta$.

The integral becomes $I = \sqrt{2}\int_{-1}^{1}\sqrt{1-\frac{x+1}{2}}\frac{\mathrm{d}x}{2} = \sqrt{2}\int_{-1}^{1}\sqrt{\frac{1-x}{2}}\frac{\mathrm{d}x}{2}$

$$= \frac{1}{2}\int_{-1}^{1}\sqrt{1-x}\ \mathrm{d}x$$

$$I = w_1 f[x_1] + w_2 f[x_2] + w_3 f[x_3]$$

$$= \frac{1}{2}\left(0.5555555 f[\text{-}0.77459667] + 0.8888889 f[0] + 0.5555555 f[0.77459667]\right)$$

$$= 0.5\left(0.5555555 \times 1.33214 + 0.8888889 \times 1 + 0.5555555 \times 0.474767\right) = \boxed{0.946363}$$

The exact answer is $\frac{2\sqrt{2}}{3} = 0.94281$

<u>QUESTION 9</u>

Consider the definite integral

$$I = \int_0^{\infty} \sqrt{(x^2+1)}\ \mathrm{e}^{-x}\ \mathrm{d}x\,.$$

The infinite upper limit does not allow for application of the Trapezoidal or Simpson's rule. Proceed as follows: Break the integral into the sum of two integrals

$$I = \int_0^{1} \sqrt{(x^2+1)}\ \mathrm{e}^{-x}\ \mathrm{d}x + \int_1^{\infty} \sqrt{(x^2+1)}\ \mathrm{e}^{-x}\ \mathrm{d}x$$

and change the variable in the second integral by substituting $y = \frac{1}{x}$. Apply Simpson's rule by choosing an h small enough to guarantee 3-decimal places accuracy.