FREDERICK UNIVERSITY CYPRUS

AMAT 314
Numerical Methods

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HOMEWORK ASSIGNMENT ON NUMERICAL SOLUTION OF NONLINEAR EQUATIONS

<u>Question 1</u>

You are required to fit the curve

$$y = \ln(ax) + a^{1.5x}$$

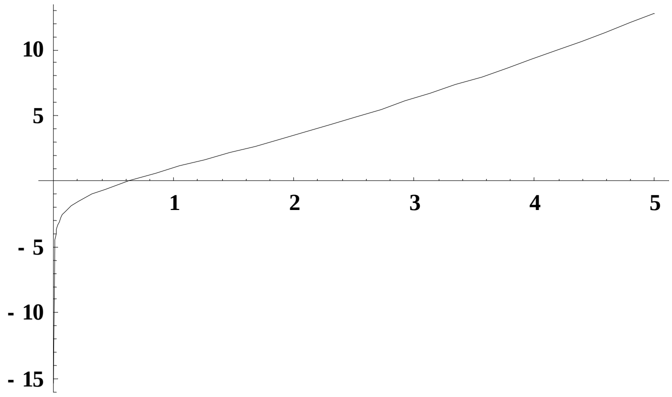
through the point $(1,0)$**.**

a. **Use a graphical method to estimate a value for** a**.**
b. **Using as an initial guess your registration number, evaluate the root correct to 2 decimal places using the Newton-Raphson method.**

In order for the curve to pass through the point (0,1) it must satisfy the equation

$$0 = \ln(a*1) + a^{1.5*1}.$$

Hence, a must satisfy the equation $\ln(a) + a^{1.5} = 0$. Plotting the function $f(a) = \ln(a) + a^{1.5}$ suggests that the root of the equation is approximately $a \approx 0.6$.



To apply the Newton-Raphson method we define the function

$$f(a) = \ln(a) + a^{1.5}$$

and its derivative

$$f'(a) = \frac{f(a)}{\mathrm{d}\,a} = \frac{1}{a} + 1.5\,a^{0.5}$$

Hence, Newton-Raphson method is as follows:

$$a_{n+1} = a_n - \frac{f(a_n)}{f'(a_n)}$$

Starting with the value $a_0 = 2500$

$$a_1 = 2500 - \frac{f(2500)}{f'(2500)} = 2500 - \frac{\ln(2500) + 2500^{1.5}}{\frac{1}{2500} + 1.5 * 2500^{0.5}}$$

$$= 2500 - \frac{125008}{75.0004} = 833.227$$

$$a_2 = 833.23 - \frac{f(833.23)}{f'(833.23)} = 833.23 - \frac{\ln(833.23) + 833.23^{1.5}}{\frac{1}{833.23} + 1.5 * 833.23^{0.5}}$$

$$= 833.23 - \frac{24058.5}{43.3} = 267.6$$

$$a_3 = 267.6 - \frac{f(267.6)}{f'(267.6)} = 267.6 - \frac{4383.12}{24.5415} = 88.9997$$

$a_4 = 29.3967,\ a_5 = 9.46655,\ a_6 = 2.8206,\ a_7 = 0.811348,\ a_8 = 0.609399,$

$a_9 = 0.616355,\ a_{10} = 0.61637$

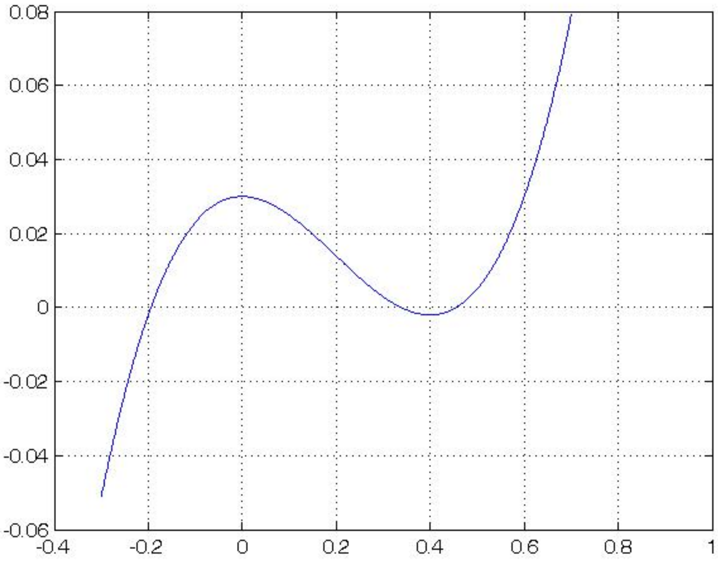
Question 2

Consider the following equation:

$$y_2^3 - 0.6\, y_2^2 = -0.03.$$

Starting with the two initial guesses $y_2 = 0$ **and** $y_2 = 0.3$ **proceed to compute three iterations using the bisection method. Keep five decimal places in all your calculations.**

A plot of the function $y = f(x) = x^3 - 0.6\, x^2 + 0.03$ is as follows:



This can be easily generated in Matlab using the commands:

```
>> x=-0.3:0.001:0.7;
>> y=x.^3-0.6*x.^2+0.03;
>> plot(x,y)
>> grid on
```

There are 3 roots $x \approx -0.2,\ 0.35,\ 0.45.$.

Improved x	x	$y = f(x) = x^3 - 0.6\,x^2 + 0.03$
Initial Guess	0.35	-0.000625
Initial Guess	0.34	-5.6000e-005
Guess Again !!!!!!	0.33	5.9700e-004
$\frac{0.33+0.34}{2} =$	0.335	2.6038e-004
$\frac{0.335+0.34}{2} =$	0.3375	9.9609e-005
$\frac{0.3375+0.34}{2} =$	0.3388	2.1154e-005

The exact answer can be found using the command ‘root’ in Matlab:

```
>> roots([1  -0.6  0  0.03])
ans =
    0.4552
    0.3391
   -0.1943
>>
```

<u>Question 3</u>

Water in a rectangular channel flows into a gradual contraction section as is indicated in Figure 1. If the flowrate is $Q = 0.708\ m^3/s$ and the upstream depth is $y_1 = 0.61\ m$, determine the downstream depth, y_2.

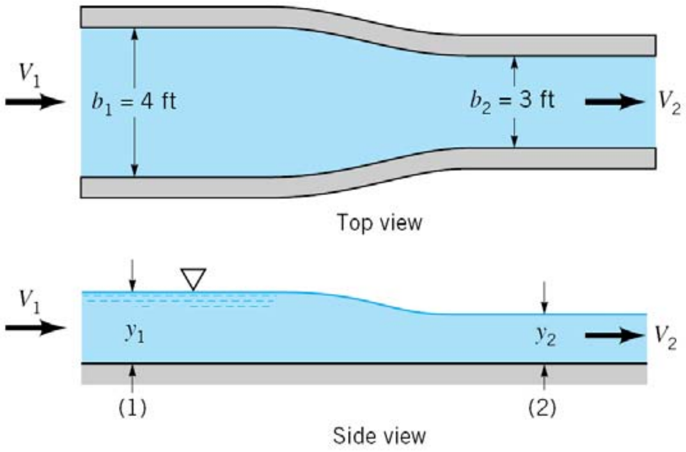


Figure 1: Top and side view of the channel

When one attempts to solve the above problem will result to the following equation:

$$y_2^3 - 0.656\ y_2^2 = -0.031\ .$$

Starting with the two initial guesses y_2=0 and y_2=0.5 proceed to compute three iterations using the bisection method. Keep five decimal places in all your calculations.

Improved x	x	$y = f(x) = x^3 - 0.656\ x^2 + 0.031$
Initial Guess	0.0	0.031
Initial Guess	0.5	-0.008
$\frac{0.0+0.5}{2} =$	0.25	0.005625
$\frac{0.5+0.25}{2} =$	0.375	-0.00851562
$\frac{0.25+0.375}{2} =$	0.3125	-0.00254492
$\frac{0.25+0.3125}{2} =$	0.28125	0.00135669

Improved x	x	$y = f(x) = x^3 - 0.656\,x^2 + 0.031$
Initial Guess	0.5	-0.008
Initial Guess	1.0	0.375
$\frac{0.5+1.0}{2}=$	0.75	0.083875
$\frac{0.5+0.75}{2}=$	0.625	0.0188906
$\frac{0.5+0.625}{2}=$	0.5625	0.00141602
$\frac{0.5+0.5625}{2}=$	0.53125	-0.00420776
$\frac{0.5625+0.53125}{2}=$	0.546875	-0.00163626

There are three solutions of the equation $x^3 - 0.656\,x^2 + 0.031 = 0$. These are $x =$ -0.191279, 0.291717 and 0.555563 correct to 6 decimal places.

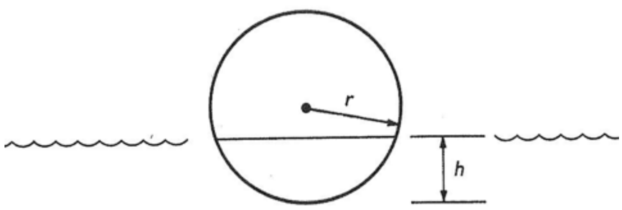
Question 4

Using a graphical method estimate a root of the equation $e^x - 3x = 0$. Using two appropriate initial guesses for x use the bisection method to find this root. How many iterations would you need to evaluate the root correct to four significant places – that is $|x - x_0| < 0.00005$. How many for eight places?

Note: The assignment involves the development of a small program-code in Matlab that applies the bisection method. I emphasize the necessity of good programming techniques.

Question 5

A sphere of radius r floats in water. The volume of a spherical segment is $\frac{\pi}{3}(3rh^2 - h^3)$. Find the depth to which a sphere of density 800 kg/m^3 sinks in water as a fraction of its radius (see figure) using the bisection and the false position method.



Applying Archimede's principle we obtain:

$$\frac{4}{3}\pi r^3 \rho_{\text{sphere}} g = \frac{\pi}{3}(3rh^2 - h^3)\rho_{H_2O} g$$

which simplifies to

$$4r^3 800 = (3rh^2 - h^3)1000 \Rightarrow 3.2r^3 = 3rh^2 - h^3$$

$$\Rightarrow h^{*3} - 3h^{*2} + 3.2 = 0 \text{ where } h^* = \frac{h}{r}$$

Using the Newton-Raphson Method

Using physical arguments or by plotting the function $f(h^*) = h^{*3} - 3h^{*2} + 3.2$, observe that there is a solution between 1.0 and 2.0

$$f(h^*) = h^{*3} - 3h^{*2} + 3.2$$

$$f'(h^*) = \frac{\mathrm{d} f(h^*)}{\mathrm{d}h^*} = 3h^{*2} - 6h^*$$

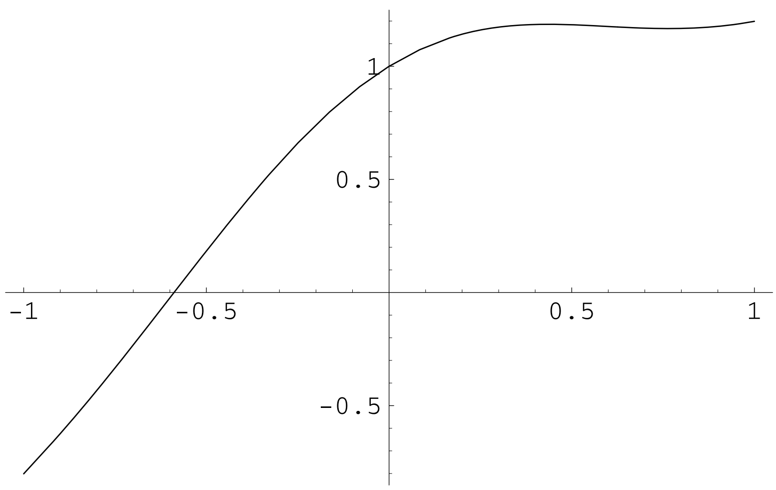
Newton-Raphson formula: $h^*_{n+1} = h^*_n - \dfrac{f(h^*_n)}{f'(h^*_n)}$

h^*_n	$f(h^*_n)$	$f'(h^*_n)$	h^*_{n+1}
1	1.2	-3	1.4
1.4	0.064	-2.52	1.425396825
1.425396825	0.00079038	-2.457112624	1.425718495
1.425718495	1.33049E-07	-2.456291289	1.425718549
1.425718549	4.09019E-10	-2.456291151	1.425718549

The exact solution is 1.425718549166519

Question 6

The plot of the function $f(x) = x + e^{-x^2}\cos(x)$ is shown in the figure below.



Using the iteration formula $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$ (Newton-Raphson process) or the bisection method determine the root of the equation

$$x + e^{-x^2}\cos(x) = 0$$

within three decimal places of accuracy, rounding your numbers to three significant places.

The figure shows the curve $y = x + e^{-x^2}\cos(x)$, hence the root of the equation $x + e^{-x^2}\cos(x) = 0$ must lie at the location $y = 0$, i.e. the point where the curve crosses the x-axis. This point lies between the points $x = -0.5$ and $x = -0.6$.

Bisection Method:
Use initial guesses $x_1 = -0.5$ and $x_2 = -0.6$.

Improved x	x	$y = x + e^{-x^2}\cos(x)$
Initial Guess	-0.5	0.1835
Initial Guess	-0.6	-0.0242
$\frac{-0.5+(-0.6)}{2} =$	-0.55	0.08
$\frac{-0.6+(-0.55)}{2} =$	-0.575	0.0279
$\frac{-0.6+(-0.575)}{2} =$	-0.5875	0.0019
$\frac{-0.6+(-0.5875)}{2} =$	-0.5938	-0.0112
$\frac{-0.5938+(-0.5875)}{2} =$	-0.5907	-0.0047
$\frac{-0.5907+(-0.5875)}{2} =$	-0.5891	-0.0015
$\frac{-0.5891+(-0.5875)}{2} =$	-0.5883	0.0002
$\frac{-0.5891+(-0.5883)}{2} =$	-0.5887	

The solution is closer to -0.6 rather than -0.5 which is not surprising since if you look closely at the graph the curve crosses the x-axis close to -0.6. Hence, it would have been more clever (and faster) to chose the initial guesses $x_1 = -0.55$ and $x_2 = -0.6$. (Από `νεσιη νουν, εσιη πόθκια).

Newton-Raphson Method:
Again if one uses as an initial guess $x_0 = -0.6$, the answer will converge faster. However, let me act stupid and use $x_0 = -0.5$.

The derivative of the function $f(x) = x + e^{-x^2}\cos(x)$ **is**

$$\frac{\mathrm{d}f(x)}{\mathrm{dx}} = 1 + (-2x)e^{-x^2}\cos(x) + e^{-x^2}(-\sin(x)) = 1 - e^{-x^2}\left(2x\cos(x) + \sin(x)\right).$$

Hence

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = x_0 - \frac{x_0 + e^{-x_0^2}\cos(x_0)}{1 - e^{-x_0^2}\left(2x_0\cos(x_0) + \sin(x_0)\right)} =$$

$$= (-0.5) - \frac{(-0.5) + e^{-(-0.5)^2}\cos(-0.5)}{1 - e^{-(-0.5)^2}\left(2(-0.5)\cos(-0.5) + \sin(-0.5)\right)} = -0.5 - \frac{0.1835}{2.0568} = -0.5892$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = x_1 - \frac{x_1 + e^{-x_1^2}\cos(x_1)}{1 - e^{-x_0^2}\left(2x_1\cos(x_1) + \sin(x_1)\right)} =$$

$$= (-0.5892) - \frac{(-0.5892) + e^{-(-0.5892)^2}\cos(-0.5892)}{1 - e^{-(-0.5892)^2}\left(2(-0.5892)\cos(-0.5892) + \sin(-0.5892)\right)} = -0.5892 - \frac{-0.0017}{2.0851} = -0.5884$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = -0.5884 - \frac{-7.23 * 10^{-9}}{2.0850} = -0.5884$$

<u>Question 7</u>

The periods of the output waveform of a 555 Timer Circuit are obtained by $T_1 + T_2 = \frac{1}{f}$**, where** f **is the frequency. Given that** $R_A = 8670$**,** $C = 0.01\times10^{-6}$ **and** $T_2 = 1.4\times10^{-4}$ **find:**

a. T_1**,** f **and the duty cycle:**

$T_1 = R_A C\ln(2)$

$$Duty\ cycle = \frac{T_1}{T_1 + T_2}\times\ 100\%.$$

b. **Write a computer program in Matlab in order to find** R_B **using the secant method:**

$$T_2 = \frac{R_A R_B C}{R_A + R_B} * \ln\left(\left|\frac{R_A - 2R_B}{2R_A - R_B}\right|\right).$$

a.

$$R_A * C * \ln(2) + T_2 = \frac{1}{f}$$

$$\rightarrow (8670) * (0.01E-6) * \ln(2) + (1.4E-4) = \frac{1}{f}$$

$$\rightarrow (6.00958E-5) + (1.4E-4) = \frac{1}{f}$$

$$\rightarrow \frac{1}{f} = 2.000958E-4$$

$$\rightarrow f = 4997.606$$

$$T_1 = R_A * C * \ln(2)$$

$$\rightarrow T_1 = (8670) * (0.01E-6) * \ln(2)$$

$$\rightarrow T_1 = 6.00958E-5$$

$$\text{Duty cycle} = \frac{T_1}{T_1 + T_2} * 100\%$$

$$\rightarrow \text{Duty cycle} = \frac{(6.00958E-5)}{(6.00958E-5) + (1.4E-4)} * 100\%$$

$$\rightarrow \text{Duty cycle} = 30.035$$

b.

Matlab M-files

File fun.m

```
function out = fun(w)

RA=8670;
C=0.01*10^(-6);
T2=1.4*10^(-4);
out=RA.*w*C/(RA+w).*log(abs((RA-2.*w)./(2*RA-w)))-T2;
```

File secant.m

```
% Plot the function to determine the points where
% it crosses the x-axis
RB=100:100:100000;
plot(RB,fun(RB))

% The following pairs of points were determined
x(1)=10000;20000;
x(2)=11000;21000;

% Secant method
i=2;
while x(i) ~= x(i-1); % continue the iteration
                      % until machine precision is achieved
    x(i+1) = x(i)- ...
          fun(x(i))*(x(i)-x(i-1))/(fun(x(i))-fun(x(i-1)));
    x(i+1)
    i=i+1;
end

%plot(x,'+')
```