APPENDIX A: DETERMINATION OF THE AREA BENEATH A CURVE

We sometimes need to evaluate the area enclosed below a curve that has been established by experiment. Suppose that such a curve, as shown in Figure A.1, has ordinates y_0, y_1, $y_2 \ldots y_n$, spaced at equal intervals h over the range from zero to L in the x-direction. The true area under the curve is

$$A = \int_{o}^{L} y\,dx$$

We aim to approximate to this result in terms of the ordinates y_0, y_1, y_2, ... y_n, as measured from the curve.

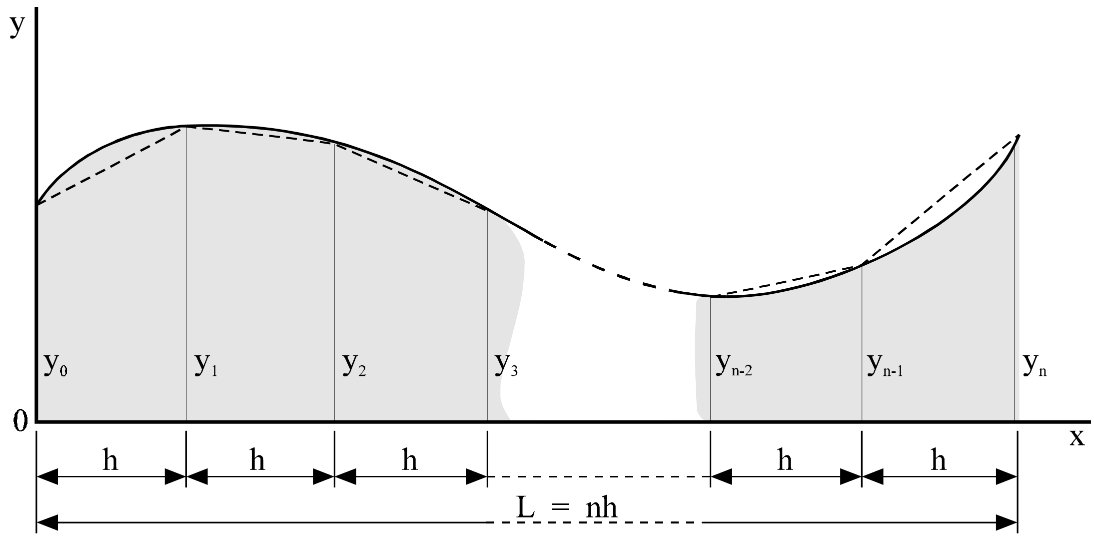

Figure A.1

The **Trapezoidal Rule** provides the simplest approximation. Imagine the curve to be replaced by the straight lines shown dashed in Figure A.1. The area of the set of trapeziums produced by this replacement is

$$A_t = \tfrac{1}{2}(y_o + y_1)h + \tfrac{1}{2}(y_1 + y_2)h + \ldots\ldots \tfrac{1}{2}(y_{n-1} + y_n)h$$

which reduces to

$$A_t = \frac{1}{2n}\left[y_0 + 2(y_1 + y_2 + \ldots\ldots + y_{n-1}) + y_n\right]L$$

The trapezoidal rule introduces error, which is obviously the sum of the differences between the areas enclosed by the curve and by the trapezoids. A much smaller error is given by **Simpson's Rule**. This gives the area enclosed under the curve between three successive points such as y_0, y_1, and y_2 as*

$$A_s = \frac{1}{3}\left(y_o + 4y_1 + y_2\right)h$$

Applying this rule repeatedly over the whole set of ordinates from y_o to y_n gives the result

$$A_s = \frac{1}{3}(y_o + 4y_1 + y_2)h + \frac{1}{3}(y_2 + 4y_3 + y_4)h + \ldots \frac{1}{3}(y_{n-2} + 4y_{n-1} + y_n)h$$

which reduces to

$$A_s = \frac{1}{3n}\left[y_o + 4\left(y_1 + y_3 + \ldots y_{n-1}\right) + 2\left(y_2 + y_4 + \ldots y_{n-2}\right) + y_\eta\right]L$$

This repeated application of Simpson's rule obviously involves an even number of intervals, i.e. n will be an even number. If, for some reason, it is necessary to divide the length L into an odd number of intervals, then Simpson's rule may be used up to the penultimate interval, and the trapezoidal rule then used for the remaining last step.

For example, when analysing Figure 5.7, we need the area

$$A = \int_o^\pi c_p \cos\theta \, d\theta$$

The mean of the two curves of Figure 5.7 is reproduced in Figure A.2, with the required area shown shaded. Note that part of the area, from about 35° to 90°, is to be reckoned negative. The following ordinates have been measured from the curve:

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* Simpson's rule is exact for any cubic curve of the form

$$y = a + bx + cx^2 + dx^3$$

So error arises only from term of 4th and higher orders of x. This is very much better than the trapezoidal rule, which is exact only to 1st order of x.

θ (°)	0	20	40	60	80	100	120	140	160	180
$c_p\cos\theta$	0.98	0.62	–0.18	–0.52	–0.17	0.16	0.47	0.72	0.88	0.92

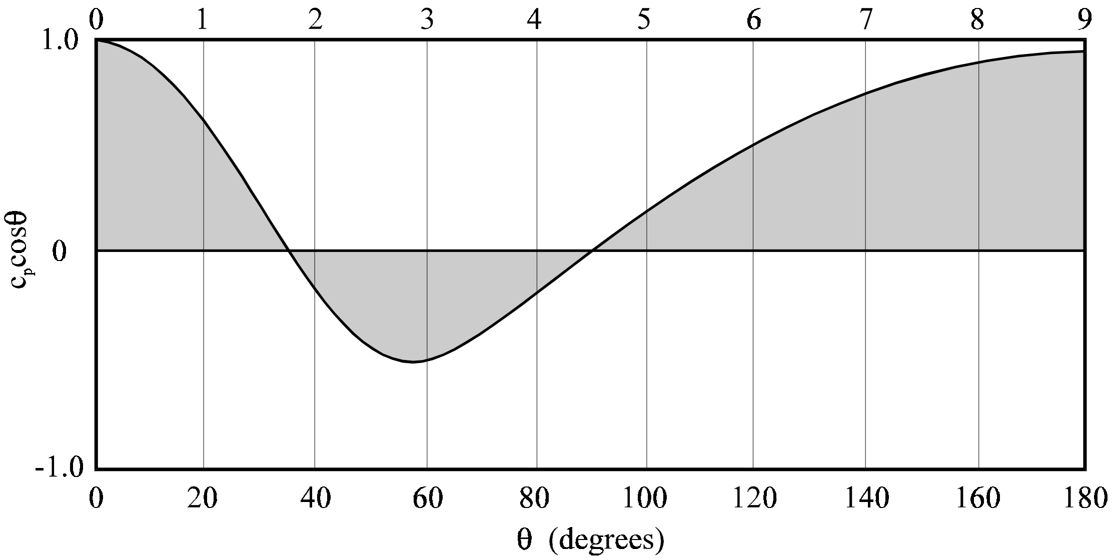

Figure A.2

Since the number of steps is odd, Simpson's rule is used in the range 0° to 160°, and the trapezoidal rule for the last step from 160° to 180°.

For 0° to 160°:

Simpson's rule with n = 8, L = 160 = 160 (π/180) radians:

$$A_S = \frac{1}{3\times 8}\left[0.98 + 4(0.62 - 0.52 + 0.16 + 0.72) + 2(-0.18 - 0.17 + 0.47) + 0.88\right]\times 160\left(\frac{\pi}{180}\right)$$

$$A_s = 0.70$$

(Note the negative ordinates, and the conversion factor (π/180) from degrees to radians)

For 160° to 180°:

Trapezoidal rule with n = 1, L = 20 = 20(π/180) radians:

$$A_t \;=\; \frac{1}{2\times 1}\left[0.88+0.92\right]\times 20\left(\frac{\pi}{180}\right) \;=\; 0.31$$

Total area A from 0 to 180 : A = $A_s + A_t$ = 1.01

So

$$\int_o^{\pi} c_p \cos\theta \, d\theta \;=\; 1.01$$

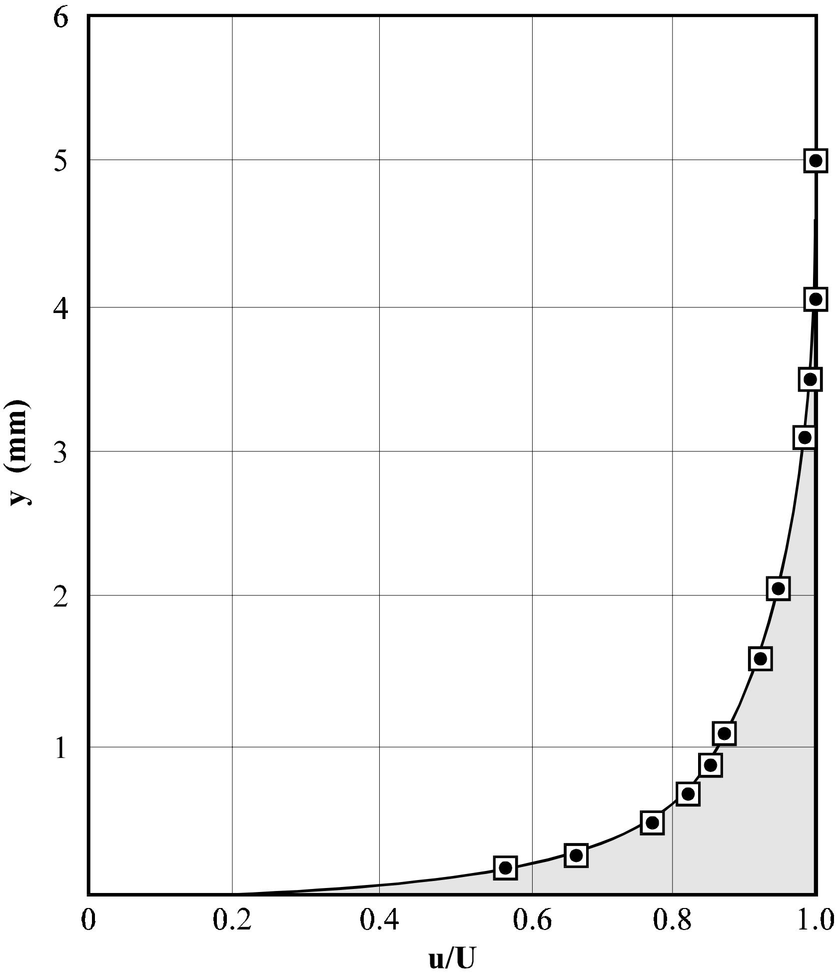

Figure A.3

As a further example, consider Figure 7.5, reproduced as Figure A.3, from which it is required to evaluate the integral:

$$\delta^* \;=\; \int_o^{\infty} \left(1-u/U\right) dy$$

This is represented by the shaded area. Note that (1 – u/U) is dimensionless, and y has dimensions of length, being measured in units of mm. So the shaded area will also appear in units of mm.

Values of (1 – u/U) read from the curve are as follows:

y (mm)	0	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5	5.0	>5.0
(1–u/U)	1.00	0.25	0.15	0.10	0.06	0.04	0.03	0.02	0.01	0.01	0.00	0.00

Simpson's rule with n = 10, L = 5.0 mm:

$$A_S = \frac{1}{3\times 10}\begin{bmatrix} 1.00 + 4(0.25 + 0.10 + 0.04 + 0.02 + 0.01) \\ + 2(0.15 + 0.06 + 0.03 + 0.01) + 0.00 \end{bmatrix} \times 5.0 \text{ mm}$$

$$A = 0.53 \text{ mm}$$

So

$$\delta^* = \int_0^\infty (1 - u/U)dy = \int_0^5 (1 - u/U)dy = 0.53 \text{ mm}$$