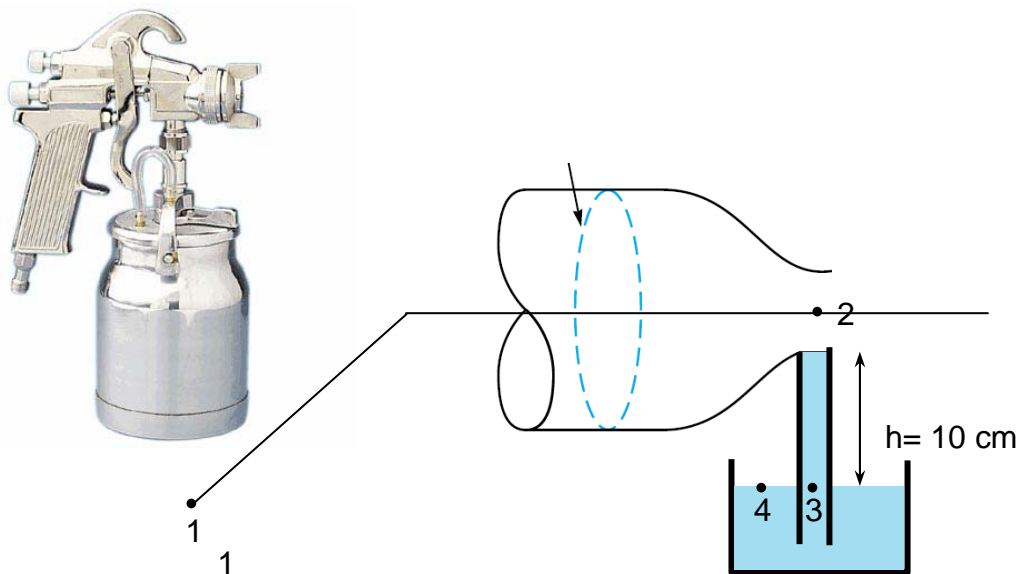


HOMWORK ASSIGNMENT ON BERNOULLI'S EQUATION

1. Conventional spray-guns operate by achieving a low pressure through high speed air passing through a nozzle (see figure) that pumps the liquid (paint) up into the stream of air. In an experiment, a compressor is used to supply air at 3 kPa (gage). Determine the velocity of the air through the nozzle if the water in the container is elevated 10 cm .



Point 1 is the compressor tank where the pressure is 3000 Pa.

Take Bernoulli's equation between point 1 and 2.

$$p_1 + \underbrace{\frac{1}{2}\rho u_1^2}_{\approx 0} + \underbrace{\rho g h_1}_{\approx 0} = p_2 + \frac{1}{2}\rho u_2^2 + \underbrace{\rho g h_2}_{\approx 0}$$

Because of the small density of air the gravity terms are negligible. Also the velocity in the compressor tank is ≈ 0 .

Hence

$$p_1 = p_2 + \frac{1}{2}\rho u_2^2.$$

Because the streamlines are parallel, only the hydrostatic pressure need to be incorporated to relate the pressures between sections 2 and 3.

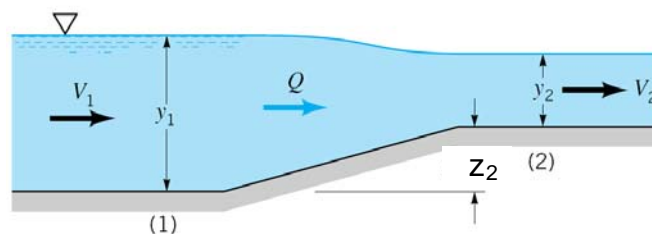
$$p_2 + \rho g h = p_3 \text{ (neglecting small pressure variation across the air stream)}$$

But $p_3 = p_{atm}$ and combining above equations we obtain

$$p_1 = p_{atm} - \rho g h + \frac{1}{2}\rho u_2^2 \Rightarrow u_2 = \sqrt{\frac{2(p_1 - p_{atm} + \rho g h)}{\rho}}$$

$$= \sqrt{\frac{2(3000 + 1000 \cdot 9.81 \cdot 0.1)}{1000}} = 2.82 \text{ m/s}$$

2. Water flows up ramp in a constant width rectangular channel at a rate $Q = 0.534 \text{ m}^3/\text{s}$. The upstream depth is $y_1 = 0.7 \text{ m}$ and the width of the channel is $b = 1 \text{ m}$.
- Write the Bernoulli's equation for a streamline between sections 1 and 2. Show the streamline
 - Write the mass conservation for sections 1 and 2.
 - If the velocity downstream $V_2 = 0.79 \text{ m/s}$ determine y_2 and z_2 .



Consider Bernoulli Equation on the streamline joining the water interface at point 1 and point 2:

$$(y_1 + z_1) + \frac{V_1^2}{2g} = (y_2 + z_2) + \frac{V_2^2}{2g} \quad \text{Eq. 1}$$

Mass conservation

$$\dot{m}_1 = \dot{m}_2 \Rightarrow \rho_1 Q_1 = \rho_2 Q_2 \text{ (incompressible)} \Rightarrow Q_1 = Q_2 = Q$$

$$\Rightarrow V_1 \cdot y_1 \cdot b = V_2 \cdot y_2 \cdot b = Q \quad \text{Eq. 2}$$

$$V_1 = \frac{Q_1}{y_1 \cdot b} = \frac{0.534}{0.7 \cdot 1} = 0.76 \text{ m/s} \quad \text{Eq. 3}$$

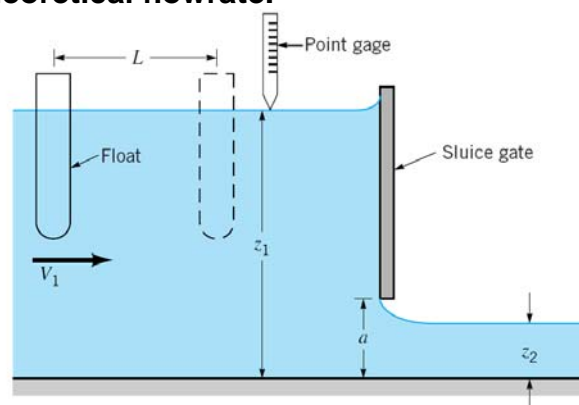
$$Q_2 = y_2 \cdot b \cdot V_2 \Rightarrow y_2 = \frac{Q_2}{b \cdot V_2} = \frac{0.534}{1 \cdot 0.79} = 0.68 \text{ m}$$

Assuming that the datum is at the level z_1 , i.e. $z_1 = 0$ and substituting eq. 2 and eq. 3 into eq. 1 we obtain

$$0.7 + 0 + \frac{0.76^2}{2 \cdot 9.81} = 0.68 + z_2 + \frac{0.79^2}{2 \cdot 9.81}$$

$$\Rightarrow z_2 = 0.7 + 0.04 - 0.68 - 0.032 = 0.028$$

3. The flowrate of water under a sluice gate as shown in the figure is a function of the water depths upstream and downstream of the gate and the width of the sluice gate. Use the Bernoulli and continuity equations to determine the theoretical flowrate.



Consider Bernoulli Equation on the streamline joining the water interface at point 1 and point 2:

$$\left. \begin{aligned} \frac{p_1}{\rho g} + \frac{V_1^2}{2g} + z_1 &= \frac{p_2}{\rho g} + \frac{V_2^2}{2g} + z_2 \\ p_1 = p_2 = p_{atm} \end{aligned} \right\} \Rightarrow \frac{V_1^2}{2g} + z_1 = \frac{V_2^2}{2g} + z_2 \quad \text{Eq. 1}$$

Mass conservation

$$\dot{m}_1 = \dot{m}_1 \text{ (incompressible)} \Rightarrow Q_1 = Q_2$$

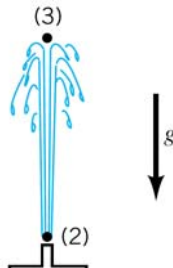
$$V_1 \cdot z_1 \cdot b = V_2 \cdot z_2 \cdot b \text{ (same width } b) \Rightarrow V_2 = \frac{V_1 \cdot z_1}{z_2}$$

$$\text{Substitute in eq.1} \Rightarrow \frac{V_1^2}{2g} + z_1 = \left(\frac{V_1 \cdot z_1}{z_2} \right)^2 \frac{1}{2g} + z_2$$

$$\Rightarrow \frac{V_1^2}{2g} \left(1 - \left(\frac{z_1}{z_2} \right)^2 \right) + (z_1 - z_2) = 0 \Rightarrow V_1 = \sqrt{\frac{2g(z_2 - z_1)}{\left(1 - (z_1/z_2)^2 \right)}}$$

$$\text{Volumetric flowrate } Q = A_1 \cdot V_1 = \sqrt{\frac{2g(z_2 - z_1)}{\left(1 - (z_1/z_2)^2 \right)}} \cdot z_1 \cdot b$$

4. A vertical nozzle projects a jet of water 50 m into the air. Use Bernoulli's equation to calculate the water velocity at the exit from the nozzle.

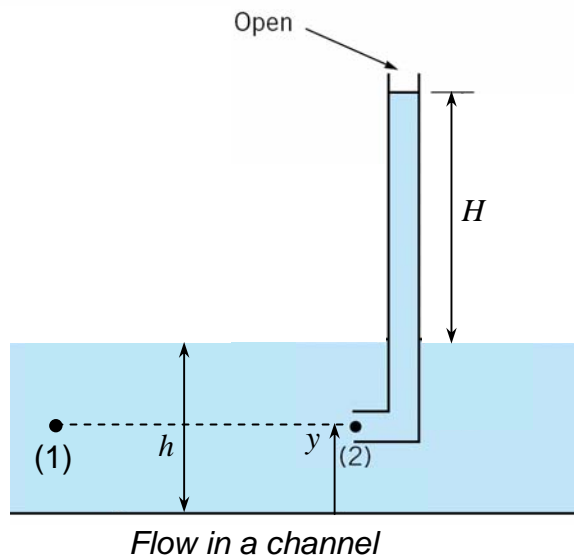


Take Bernoulli's equation between point 2 and 3.

$$\underbrace{p_2}_{=p_{atm}} + \frac{1}{2} \rho u_2^2 + \underbrace{\rho g h_2}_{=0} = \underbrace{p_3}_{=p_{atm}} + \frac{1}{2} \rho \underbrace{u_3^2}_{\approx 0} + \rho g h_3$$

$$\Rightarrow \frac{1}{2} \rho u_2^2 = \rho g h_3 \Rightarrow u_2 = \sqrt{2gh_3} = 31.4 \text{ m/s}$$

5. In a rectangular channel the velocity is measured using a pitot tube as shown on Figure 4. Based on the measurements it was concluded that the height (H) is related to the depth (y) by the relation $H = 0.2y^2$. Determine:
- The velocity profile and the shear stress.
 - If the cross-sectional area has dimensions 50×50 (cm) determine the volumetric flow rate.
 - What second order correction should be included to the velocity equation in order to obtain a no shear stress condition on the surface $y = h$.



$$\left. \begin{array}{l} p_1 + \frac{1}{2} \rho u_1^2 + \rho g h_1 = p_2 + \frac{1}{2} \rho u_2^2 + \rho g h_2 \\ h_1 = h_2 \\ u_2 \approx 0 \end{array} \right\} \Rightarrow p_2 = p_1 + \frac{1}{2} \rho u_1^2$$

The hydrostatic pressure at points (1) and (2) can be obtained from statics:

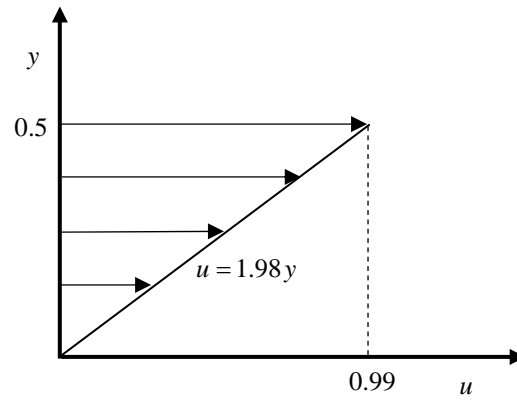
$$\begin{aligned} p_1 &= \rho g(h - y) + p_{atm} \\ p_2 &= \rho g(H + h - y) + p_{atm} \end{aligned}$$

Combine all three equations to obtain:

$$\rho g(H + h - y) + p_{atm} = \rho g(h - y) + p_{atm} + \frac{1}{2} \rho u_1^2$$

Above equations can be simplified to obtain

$$\left. \begin{array}{l} u_1^2 = 2gH \\ \text{but } H = 0.2y^2 \end{array} \right\} \Rightarrow u_1^2 = 2g \times 0.2y^2 \Rightarrow u_1 = 1.98y$$



$$Q = \text{area of triangular prism} = \frac{0.99 \times 0.5}{2} \times 0.5 = 0.124 \text{ m}^3/\text{s}$$

or Integrate

$$Q = \int u \, dA = \int u \, b \, dy = \int_0^{0.5} 1.98y \times 0.5 \, dy = 1.98 \times 0.5 \left. \frac{y^2}{2} \right|_0^{0.5} = 1.98 \times 0.5^4 = 0.124 \text{ m}^3/\text{s}$$

$$u = 1.98y + ay^2$$

$$\frac{du}{dy}(y = 0.5) = 0 \Rightarrow 1.98 + 2ay = 1.98 + 2a \times 0.5 \Rightarrow a = 1.98$$

6. Figure shows a pump that draws a liquid of density 920 kg/m^3 from a sealed underground tank. In order to prime the pump with liquid prior to a pumping operation, the delivery valve is closed and a compressed air supply at a stagnation pressure of 5 bars is allowed to flow through the reducer shown in the figure. The tank and the pump body are connected as indicated on the figure. For the conditions shown, assume incompressible flow through the reducer and calculate the maximum values of h for which priming can be achieved.

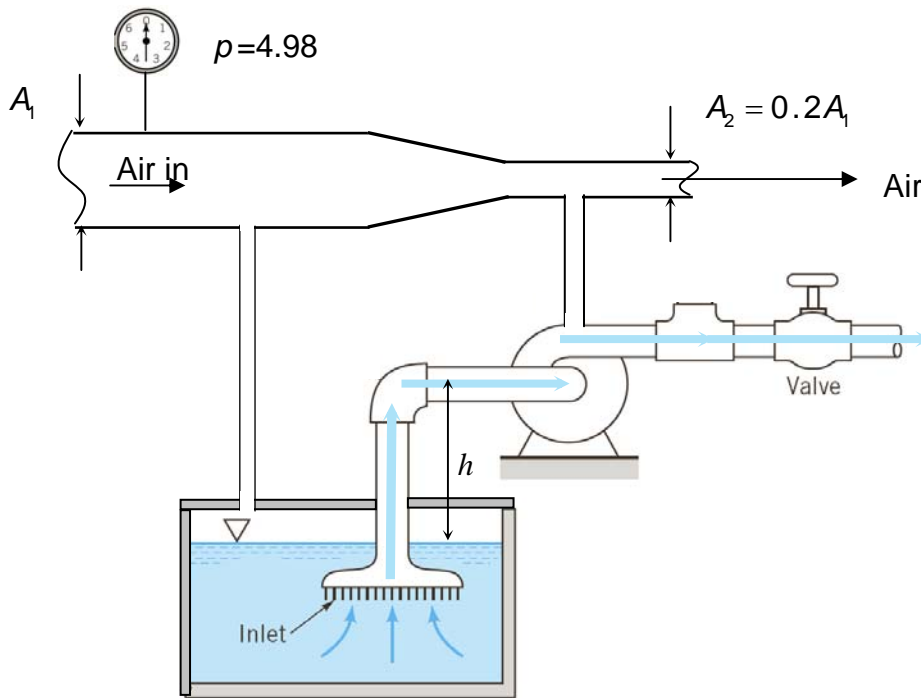


Figure 5: Reducer-nozzle

Take Bernoulli's equation between points 1 and 2.

$$\underbrace{p_1}_{=498000} + \frac{1}{2} \rho u_1^2 + \underbrace{\rho g h_1}_{\approx 0} = p_2 + \frac{1}{2} \rho u_2^2 + \underbrace{\rho g h_2}_{\approx 0} = 500000$$

Mass conservation between points 1 and 2

$$\rho_1 u_1 A_1 = \rho_2 u_2 A_2 \text{ assume incompressible } \Rightarrow u_1 A_1 = u_2 A_2$$

$$\Rightarrow u_1 A_1 = 0.2 u_2 A_1 \Rightarrow u_1 = 0.2 u_2$$

Hydrostatic pressure between points 1 and 2

$$p_2 + \rho_l g h = p_1$$

From Bernoulli's at point 1 and mass conservation

$$\underbrace{p_1}_{=498000} + \frac{1}{2} \rho (0.2u_2)^2 = 500000 \Rightarrow \frac{1}{2} \rho u_2^2 = \frac{500000 - 498000}{0.2^2} = 50000$$

From above and Bernoulli's equation at point 2

$$p_2 = 500000 - \frac{1}{2} \rho u_2^2 = 500000 - 50000 = 450000$$

From above and hydrostatic pressure equation

$$h = \frac{p_1 - p_2}{\rho_l g} = \frac{498000 - 450000}{920 \cdot 9.81} = 5.32 \text{ m}$$

7. Air is drawn in the wind tunnel used for testing automobiles. Determine the manometer reading, h , when the velocity in the test section is 60 km/hr. Determine the difference between the stagnation pressure on the front of the automobile and the pressure in the test section.

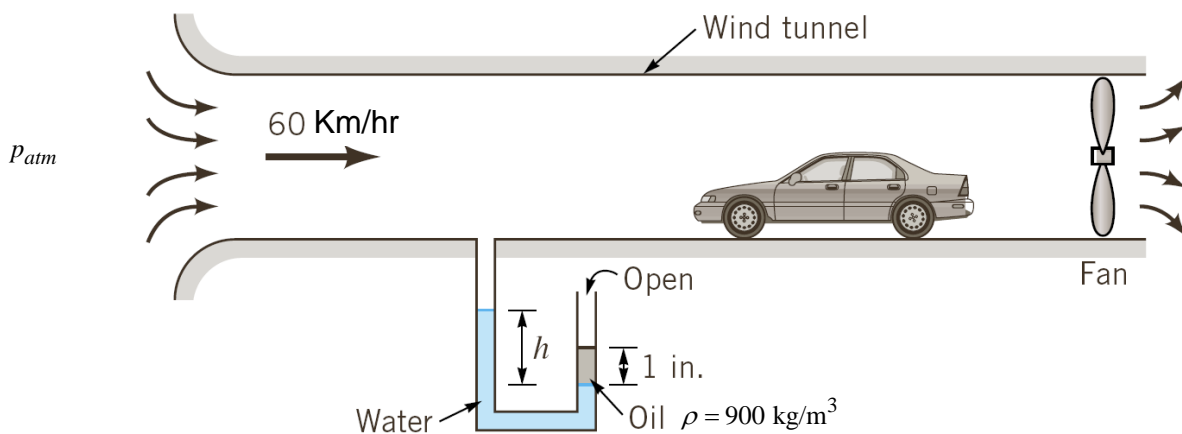


Figure 5: Wind tunnel (Question 1)

$$p_{atm} = p_{section} + \frac{1}{2} \rho_{air} \left(\frac{60}{3.6} \right)^2$$

$$p_{section} + \rho_{water} g h = p_{atm} + \rho_{oil} g \cdot 0.0254$$

8. A hovercraft (air cushion vehicle) is supported by forcing air into the chamber created by a skirt around the periphery of the vehicle as shown in Figure 7. The air escapes through the 10 cm clearance between the lower end of the skirt and the ground (or water). Assume the vehicle weighs 5000 kg and is essentially rectangular in shape, 10 by 20 m. The volume of the chamber is large enough so that the kinetic energy of the air within the chamber is negligible. Determine the flowrate, Q , needed to support the vehicle. If the ground clearance were reduced to 4 cm, what flowrate would be needed? If the vehicle weight were reduced to 2500 kg and the ground clearance maintained at 10 cm, what flowrate would be needed?

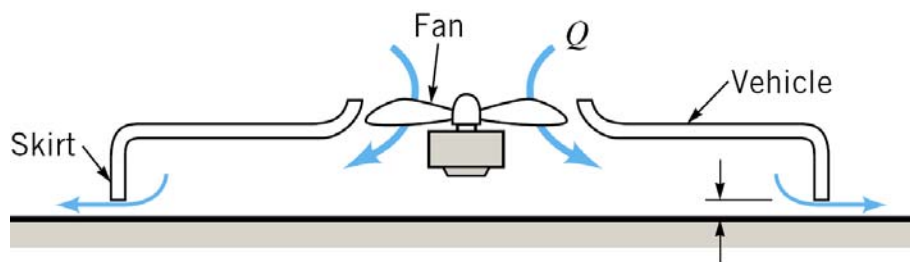


Figure : Hovercraft

The pressure inside the chamber must be greater than atmospheric, and provides a lift force

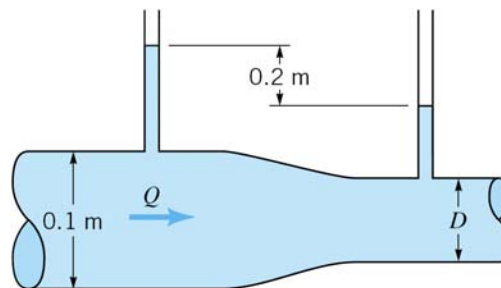
$$F = pA = mg \Rightarrow p(\text{gage}) = \frac{mg}{A} = \frac{5000 \times 9.81}{10 \times 20}$$

Take a streamline from inside the chamber to the lower end of the skirt:

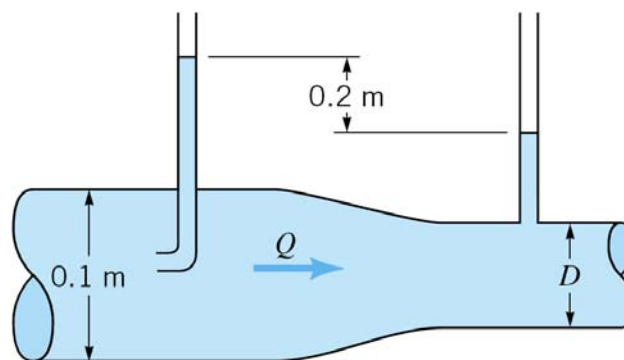
$$p_{\text{atm}} + \frac{1}{2} \rho u^2 = p \Rightarrow u = \sqrt{\frac{2 p(\text{gage})}{\rho}}$$

$$Q = uA = u \times ((10 + 10 + 20 + 20) \times 10)$$

9. Water flows through the pipe contraction shown in the figure. For the given 0.2-m difference in the manometer level, determine the flowrate as a function of the diameter of the small pipe, D .



10. Water flows through the pipe contraction shown in the figure. For the given 0.2-m difference in the manometer level, determine the flowrate as a function of the diameter of the small pipe, D .



Water flows through the pipe contraction shown in Fig. P3.18. For the given 0.2-m difference in manometer level, determine the flowrate as a function of the diameter of the small pipe, D .

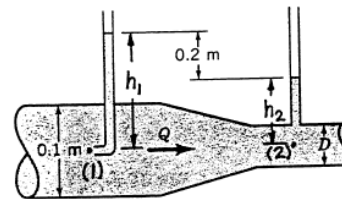


FIGURE P3.18

$$\frac{p_1}{\rho} + \frac{V_1^2}{2g} + z_1 = \frac{p_2}{\rho} + \frac{V_2^2}{2g} + z_2 \quad \text{or with } z_1 = z_2 \text{ and } V_1 = 0$$

$$V_2 = \sqrt{2g \frac{(p_1 - p_2)}{\rho}}$$

$$\text{but } p_1 = \rho h_1 \text{ and } p_2 = \rho h_2 \text{ so that } p_1 - p_2 = \rho(h_1 - h_2) = 0.2\rho$$

Thus,

$$V_2 = \sqrt{2g \frac{0.2\rho}{\rho}} = \sqrt{2g(0.2)}$$

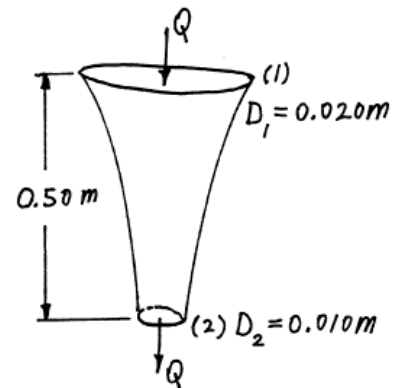
or

$$Q = A_2 V_2 = \frac{\pi}{4} D^2 V_2 = \frac{\pi}{4} D^2 \sqrt{2(9.81)(0.2)} = \underline{\underline{1.56 D^2 \frac{m^3}{s}}} \text{ when } D \sim m$$

11.



The circular stream of water from a faucet is observed to taper from a diameter of 20 mm to 10 mm in a distance of 50 cm. Determine the flowrate.



$$\frac{p_1}{\rho} + \frac{V_1^2}{2g} + z_1 = \frac{p_2}{\rho} + \frac{V_2^2}{2g} + z_2$$

where $p_1 = p_2 = 0$, $z_2 = 0$, $z_1 = 0.50 \text{ m}$
and

$$V_1 = \frac{Q}{A_1}, \quad V_2 = \frac{Q}{A_2}$$

Thus,

$$\left(\frac{Q}{A_1}\right)^2 + 2gz_1 = \left(\frac{Q}{A_2}\right)^2 \text{ or } Q = \left[\frac{2gz_1}{\left(\frac{1}{A_2^2} - \frac{1}{A_1^2}\right)} \right]^{\frac{1}{2}} = \frac{A_2 \sqrt{2gz_1}}{\sqrt{1 - (A_2/A_1)^2}}$$

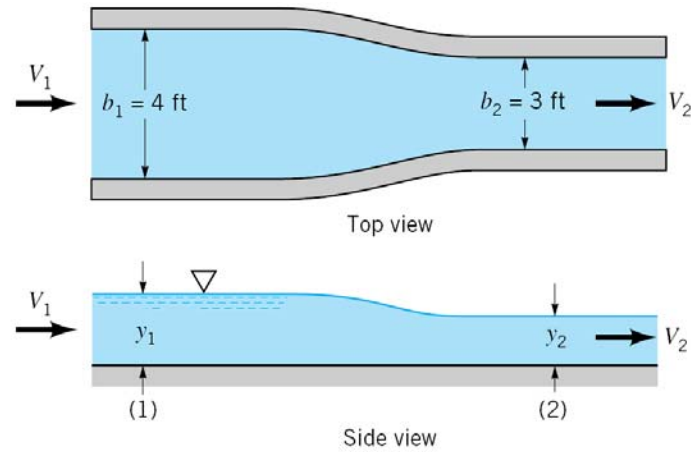
or since

$$\frac{A_2}{A_1} = \left(\frac{D_2}{D_1}\right)^2 \text{ we obtain}$$

$$Q = A_2 \frac{\sqrt{2gz_1}}{\sqrt{1 - (D_2/D_1)^4}} = \frac{\pi}{4} (0.010 \text{ m})^2 \left[\frac{2(9.81 \frac{\text{m}}{\text{s}^2})(0.50 \text{ m})}{1 - \left(\frac{0.010}{0.020}\right)^4} \right]^{\frac{1}{2}}$$

$$= \underline{\underline{2.54 \times 10^{-4} \frac{\text{m}^3}{\text{s}}}}$$

12. Water in a rectangular channel flows into a gradual contraction section as is indicated in the figure. If the flowrate is $Q = 0.708 \text{ m}^3 / \text{s}$ and the upstream depth is $y_1 = 0.61 \text{ m}$, determine the downstream depth, y_2 .



Consider Bernoulli Equation on the streamline joining the water interface at point 1 and point 2:

$$y_1 + \frac{V_1^2}{2g} + z_1 = y_2 + \frac{V_2^2}{2g} + z_2 \quad \text{Eq. 1}$$

Mass conservation

$\dot{m}_1 = \dot{m}_2$ (incompressible) $\Rightarrow Q_1 = Q_2 = 0.708 \text{ m}^3/\text{s}$ (note that the width is not the same)

$$\Rightarrow Q_1 = V_1 \cdot y_1 \cdot b_1 = V_2 \cdot y_2 \cdot b_2 \Rightarrow V_2 = \frac{Q_1}{y_2 \cdot b_2} = \frac{0.708}{y_2 \cdot 0.914} \quad \text{Eq. 2}$$

$$V_1 = \frac{Q_1}{y_1 \cdot b_1} = \frac{0.708}{0.61 \cdot 1.22} = 0.952 \text{ m/s} \quad \text{Eq. 3}$$

Assuming that the datum is at the level z_1 , i.e. $z_1 = 0$ and substituting eq. 2 and eq. 3 into eq. 1 we obtain

$$0.61 + \frac{0.952^2}{2 \cdot 9.81} + 0 = y_2 + \left(\frac{0.708}{y_2 \cdot 0.914} \right)^2 \frac{1}{2 \cdot 9.81} + 0$$

$$0.656 = y_2 + \frac{0.031}{y_2^2}$$

Multiplying by y_2^2 and simplifying we obtain

$$y_2^3 - 0.656 \cdot y_2^2 + 0.031 = 0$$

The solutions of the above equation are

$$y_2 = -0.191, y_2 = 0.292, y_3 = 0.556$$

To determine which of the positive roots to choose we need to compare the values with the critical value.

$$\text{The specific energy head } E = y + \frac{V^2}{2g} = y + \left(\frac{Q}{y \cdot b}\right)^2 \frac{1}{2g}$$

$$\text{The minimum occurs when } \frac{dE}{dy} = 1 - \left(\frac{Q}{b}\right)^2 \frac{1}{g \cdot y_c^3} = 0$$

$$\Rightarrow y_c = \sqrt[3]{\left(\frac{Q}{b}\right)^2 \frac{1}{g}}$$

Note that because b is not the same the critical value is different at each section.

$$\text{At section 1, the critical value is } y_{c_1} = \sqrt[3]{\left(\frac{0.708}{1.22}\right)^2 \frac{1}{g}} = 0.325\text{m.}$$

Since $y_1 > y_{c_1}$, it means that the flow upstream is subcritical.

The flow will remain subcritical since the reduction in width is gradual, i.e. we will choose the biggest value, i.e. $y_2 = 0.556\text{m}$.