FREDERICK INSTITUTE OF TECHNOLOGY

Mechanical Engineering Programme of Study

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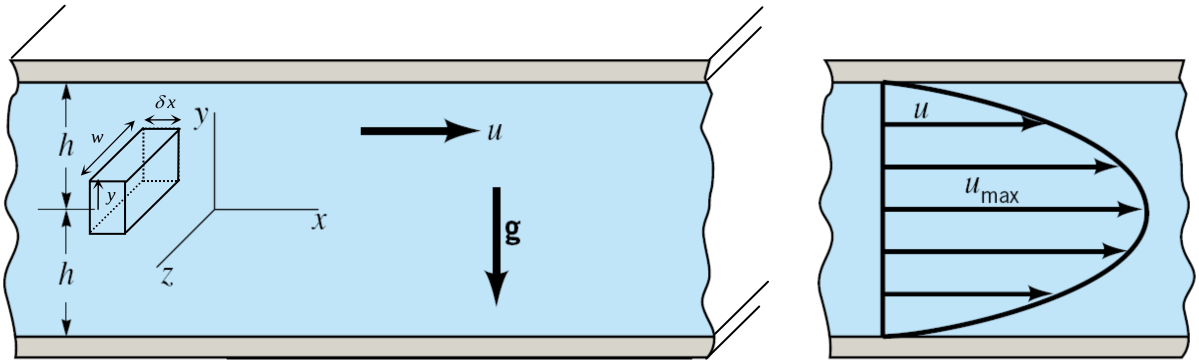
Fluid Mechanics

SOLVED EXAMPLES ON VISCOUS FLOW

1. **Consider steady, laminar flow between two fixed parallel plates due to a pressure gradient. Using a control volume of unit depth, height $2y$, and width δx (centred at $y=0$) obtain an expression for the velocity profile.**
   a. **By integrating the velocity profile obtain an expression for the volumetric flow rate and the mean velocity.**
   b. **Obtain an expression for the dimensionless pressure loss as a function of the Reynolds number.**



(a) (b)

Conservation of Momentum of the control volume

Consider x-momentum conservation

$\dot{M}_{out} - \dot{M}_{in} = \sum F_x = 0$ (steady-state so net momentum flux is zero)

The forces acting are:

i. right surface: $p_r A_r$

ii. left surface: $p_l A_l$

iii. top surface: $\tau_t A_t$

iv. bottom surface: $\tau_b A_b$

$$\left.\begin{array}{l}\text{Balance of forces: } -p_r A_r + p_l A_l = -(\tau_t A_t + \tau_b A_b) \\ \text{Because of symmetry: } \tau_t = \tau_b = \tau \\ \text{Areas are given by: } A_r = A_l = 1\cdot 2y,\ A_t = A_b = 1\cdot\delta x\end{array}\right\} \Rightarrow$$

$$\left.\begin{array}{l}2y(p_l - p_r) = -\tau\, 2\,\delta x \\ \tau = \mu \dfrac{\mathrm{d}u}{\mathrm{d}y} \text{ (Newtonian fluid)} \\ \dfrac{\mathrm{d}p}{\mathrm{d}x} = \text{constant (pressure increases linearly)}\end{array}\right\} \Rightarrow$$

$$y\left(p - (p + \delta p)\right) = -\mu \frac{\mathrm{d}u}{\mathrm{d}y}\delta x \Rightarrow y\frac{\mathrm{d}p}{\mathrm{d}x} = \mu\frac{\mathrm{d}u}{\mathrm{d}y} \Rightarrow$$

Integrate above expression

$$\mathrm{d}u = \frac{y}{\mu}\frac{\mathrm{d}p}{\mathrm{d}x}\mathrm{d}y \Rightarrow u = \int \frac{y}{\mu}\frac{\mathrm{d}p}{\mathrm{d}x}\mathrm{d}y = \frac{y^2}{2\mu}\frac{\mathrm{d}p}{\mathrm{d}x} + \text{constant}$$

To find the constant use the boundary conditions, i.e.

at $y = h$ and $y = -h$ the velocity is zero (u = 0)

$$u(y = h) = 0 = \frac{h^2}{2\mu}\frac{\mathrm{d}p}{\mathrm{d}x} + \text{constant} \Rightarrow \ \text{constant}=-\frac{h^2}{2\mu}\frac{\mathrm{d}p}{\mathrm{d}x}$$

$$\text{so u}=\frac{y^2}{2\mu}\frac{\mathrm{d}p}{\mathrm{d}x} - \frac{h^2}{2\mu}\frac{\mathrm{d}p}{\mathrm{d}x} = \frac{h^2}{2\mu}\left(-\frac{\mathrm{d}p}{\mathrm{d}x}\right)\left(1 - \left(\frac{y}{h}\right)^2\right)$$

To find the volumetric flow rate:

$$Q = \int u\ \mathrm{d}A = \int u\ 1\cdot\mathrm{d}y = \frac{h^2}{2\mu}\left(-\frac{\mathrm{d}p}{\mathrm{d}x}\right)\int_{-h}^{h}\left(1 - \left(\frac{y}{h}\right)^2\right)\mathrm{d}y = \frac{h^2}{2\mu}\left(-\frac{\mathrm{d}p}{\mathrm{d}x}\right)\left(\frac{4h}{3}\right)$$

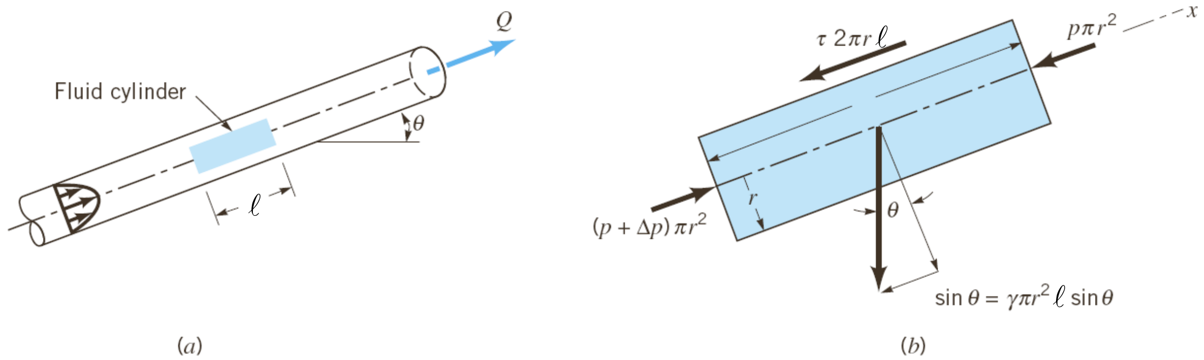
$$= \frac{2h^3}{3\mu}\left(-\frac{\mathrm{d}p}{\mathrm{d}x}\right)$$

$$\text{(mean velocity) } u_m = \frac{Q}{A} = \frac{\frac{2h^3}{3\mu}\left(-\frac{\mathrm{d}p}{\mathrm{d}x}\right)}{2h} = \frac{h^2}{3\mu}\left(-\frac{\mathrm{d}p}{\mathrm{d}x}\right)$$

b. Dimensionless pressure drop

$$\Delta p = \frac{3\ell u_m \mu}{h^2} \Rightarrow \frac{\Delta p}{\frac{1}{2}\rho u_m^2} = \frac{3\ell u_m \mu}{\frac{1}{2}\rho u_m^2 h^2} = \frac{6\ell\mu}{\rho u_m h^2} = 12\frac{\ell}{h}\frac{\mu}{\rho u_m 2h} = 24\frac{\ell}{2h}\frac{1}{\mathrm{Re}}$$

2. **Working in a similar fashion as for the case of a horizontal cylinder, obtain the velocity profile of Poiseuille's law in an inclined pipe using the control volume suggested in the figure.**



Conservation of Momentum of the control volume

Consider x-momentum conservation

$\dot{M}_{out} - \dot{M}_{in} = \sum F_x = 0$ (steady-state so net momentum flux is zero)

The force balance can be written as:

$$\left.\begin{array}{l}(p+\Delta p)\pi r^2 - p\pi r^2 - \tau 2\pi r\ell - mg\sin\theta = 0 \\ m = \pi r^2 \ell\rho\end{array}\right\} \Rightarrow$$

$$\left.\begin{array}{l}\dfrac{\Delta p - \rho g\ell\sin\theta}{\ell} = \dfrac{2\tau}{r} \\ \tau = -\mu\dfrac{\mathrm{d}u}{\mathrm{d}r}\end{array}\right\} \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}r} = -\frac{\Delta p - \rho g\ell\sin\theta}{2\ell\mu}r$$

$$u = -\frac{\Delta p - \rho g\ell\sin\theta}{2\ell\mu}\int r\mathrm{d}r = -\frac{\Delta p - \rho g\ell\sin\theta}{2\ell\mu}\frac{r^2}{2} + \text{constant}$$

Evaluate the constant using the boundary conditions:

$$u(r=R) = 0 \Rightarrow 0 = -\frac{\Delta p - \rho g\ell\sin\theta}{4\ell\mu}R^2 + \text{constant}$$

$$\Rightarrow \text{constant} = \frac{\Delta p - \rho g\ell\sin\theta}{4\ell\mu}R^2$$

$$u = \frac{(\Delta p - \rho g \ell \sin\theta)R^2}{4\ell\mu}\left(1 - \left(\frac{r}{R}\right)^2\right)$$

$$Q = \int u dA = \int_0^R u 2\pi r dr = \frac{(\Delta p - \rho g \ell \sin\theta)\pi R^2}{2\ell\mu}\int_0^R \left(1 - \left(\frac{r}{R}\right)^2\right) r dr$$

$$= \frac{(\Delta p - \rho g \ell \sin\theta)\pi R^2}{2\ell\mu}\frac{R^2}{4} = \frac{\pi(\Delta p - \rho g \ell \sin\theta)R^4}{8\ell\mu}$$

3. An oil with a viscosity of $\mu = 0.4\ \mathrm{N \cdot s/m^2}$ and density $\rho = 900\ \mathrm{kg/m^3}$ flows in a pipe of diameter $D = 0.2\ \mathrm{m}$. (a) What pressure drop $p_1 - p_2$, is needed to produce a flowrate of $Q = 2.0 \times 10^{-5}\ \mathrm{m^3/s}$ if the pipe is horizontal with $x_1 = 0$ and $x_2 = 10\ \mathrm{m}$? (b) How steep a hill, θ, must the pipe be on if the oil is to flow through the pipe at the same rate as in part (a), but with $p_1 = p_2$? (c) For the conditions of part (b), if $p_1 = 200\ \mathrm{kPa}$, what is the pressure at section $x_3 = 5\ \mathrm{m}$ where x is measured along the pipe?

(a) If the Reynolds number is less than 2100 the flow is laminar and the equations derived in this section are valid. Since the average velocity is $V = Q/A = (2.0 \times 10^{-5}\ \mathrm{m^3/s})/[\pi(0.020)^2 \mathrm{m^2}/4] = 0.0637\ \mathrm{m/s}$, the Reynolds number is $\mathrm{Re} = \rho V D/\mu = 2.87 < 2100$. Hence, the flow is laminar and from Eq. 8.9 with $\ell = x_2 - x_1 = 10\ \mathrm{m}$, the pressure drop is

$$\Delta p = p_1 - p_2 = \frac{128\mu\ell Q}{\pi D^4}$$

$$= \frac{128(0.40\ \mathrm{N \cdot s/m^2})(10.0\ \mathrm{m})(2.0 \times 10^{-5}\ \mathrm{m^3/s})}{\pi(0.020\ \mathrm{m})^4}$$

or

$$\Delta p = 20{,}400\ \mathrm{N/m^2} = 20.4\ \mathrm{kPa} \qquad \textbf{(Ans)}$$

4. **Consider steady, laminar flow in a circular pipe due to a pressure gradient. Using a control volume of length ℓ and radius r obtain an expression for the velocity profile. Follow the steps below:**
   a. **Consider the control volume below (Figure 1) and indicate the forces exerted on the control volume. Give a physical explanation.**

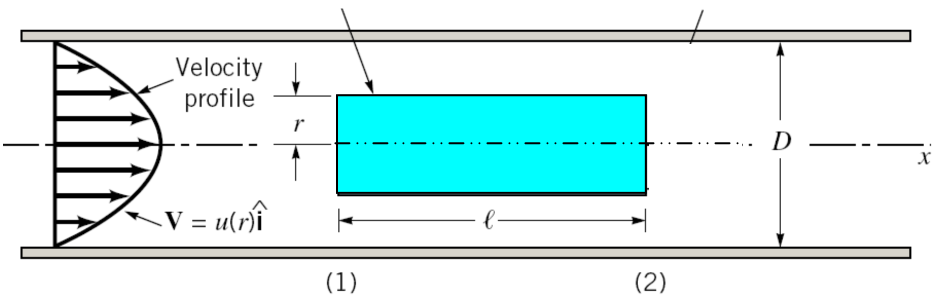


Figure 1: Laminar flow in a circular pipe.

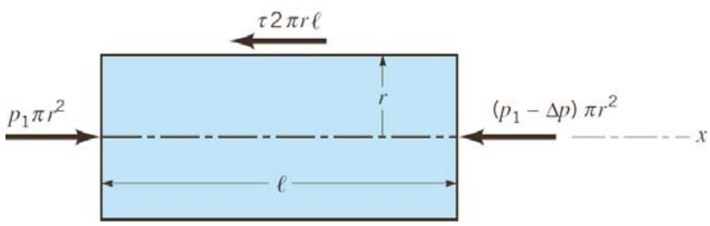
   a. **Doing a force balance show that the momentum equation can be simplified to:**

$$\frac{\Delta p}{\ell} = \frac{2\tau}{r}.$$

   c. **Assuming laminar flow of a Newtonian fluid and applying an appropriate boundary condition obtain that the velocity profile is:**

$$u = \frac{\Delta p\, D^2}{16\mu\ell}\left[1-\left(\frac{2r}{D}\right)^2\right].$$

   d. **Integrate above expression to find the volumetric flow rate.**



The forces acting on the control volume are the shear forces acting on the perimetric area $\tau 2\pi r\ell$, and pressure forces acting on the fore and aft cross-sectional areas $p\pi r^2$ and $(p-\delta p)\pi r^2$, respectively.

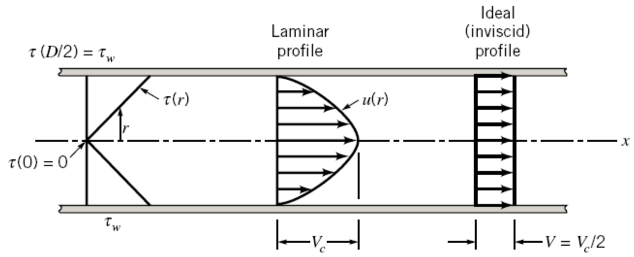
By doing a force balance $p\pi r^2 = \tau 2\pi r\ell + (p-\delta p)\pi r^2 \Rightarrow \dfrac{\delta p}{\ell} = \dfrac{2\tau}{r}$

is very complex. However, for laminar flow of a Newtonian fluid, the shear stress is simply proportional to the velocity gradient, "$\tau = \mu\, du/dy$" (see Section 1.6). In the notation associated with our pipe flow, this becomes

$$\tau = -\mu \frac{du}{dr} \tag{8.6}$$

The negative sign is included to give $\tau > 0$ with $du/dr < 0$ (the velocity decreases from the pipe centerline to the pipe wall).

Equations 8.3 and 8.6 represent the two governing laws for fully developed laminar flow of a Newtonian fluid within a horizontal pipe. The one is Newton's second law of motion and the other is the definition of a Newtonian fluid. By combining these two equations we obtain



■ **FIGURE 8.9** **Shear stress distribution within the fluid in a pipe (laminar or turbulent flow) and typical velocity profiles.**

$$\frac{du}{dr} = -\left(\frac{\Delta p}{2\mu\ell}\right) r$$

which can be integrated to give the velocity profile as follows:

$$\int du = -\frac{\Delta p}{2\mu\ell} \int r\, dr$$

or

$$u = -\left(\frac{\Delta p}{4\mu\ell}\right) r^2 + C_1$$

where C_1 is a constant. Because the fluid is viscous it sticks to the pipe wall so that $u = 0$ at $r = D/2$. Thus, $C_1 = (\Delta p/16\mu\ell)D^2$. Hence, the velocity profile can be written as

$$u(r) = \left(\frac{\Delta p D^2}{16\mu\ell}\right)\left[1 - \left(\frac{2r}{D}\right)^2\right] = V_c\left[1 - \left(\frac{2r}{D}\right)^2\right] \tag{8.7}$$

where $V_c = \Delta p D^2/(16\mu\ell)$ is the centerline velocity. An alternative expression can be written by using the relationship between the wall shear stress and the pressure gradient (Eqs. 8.5 and 8.7) to give

$$u(r) = \frac{\tau_w D}{4\mu}\left[1 - \left(\frac{r}{R}\right)^2\right]$$

where $R = D/2$ is the pipe radius.

This velocity profile, plotted in Fig. 8.9, is parabolic in the radial coordinate, r, has a maximum velocity, V_c, at the pipe centerline, and a minimum velocity (zero) at the pipe wall. The volume flowrate through the pipe can be obtained by integrating the velocity profile across the pipe. Since the flow is axisymmetric about the centerline, the velocity is constant on small area elements consisting of rings of radius r and thickness dr. Thus,

$$Q = \int u\,dA = \int_{r=0}^{r=R} u(r) 2\pi r\,dr = 2\pi\, V_c \int_0^R \left[1 - \left(\frac{r}{R}\right)^2\right] r\,dr$$

or

$$Q = \frac{\pi R^2 V_c}{2}$$

By definition, the average velocity is the flowrate divided by the cross-sectional area, $V = Q/A = Q/\pi R^2$, so that for this flow

$$V = \frac{\pi R^2 V_c}{2\pi R^2} = \frac{V_c}{2} = \frac{\Delta p D^2}{32\mu\ell} \tag{8.8}$$

and

$$\boxed{Q = \frac{\pi D^4\,\Delta p}{128\mu\ell}} \tag{8.9}$$

(b) If the pipe is on a hill of angle θ such that $\Delta p = p_1 - p_2 = 0$, Eq. 8.12 gives

$$\sin\theta = -\frac{128\mu Q}{\pi\rho g D^4} \tag{1}$$

or

$$\sin\theta = \frac{-128(0.40\ \text{N}\cdot\text{s/m}^2)(2.0\times 10^{-5}\ \text{m}^3/\text{s})}{\pi(900\ \text{kg/m}^3)(9.81\ \text{m/s}^2)(0.020\ \text{m})^4}$$

Thus, $\theta = -13.34°$. **(Ans)**

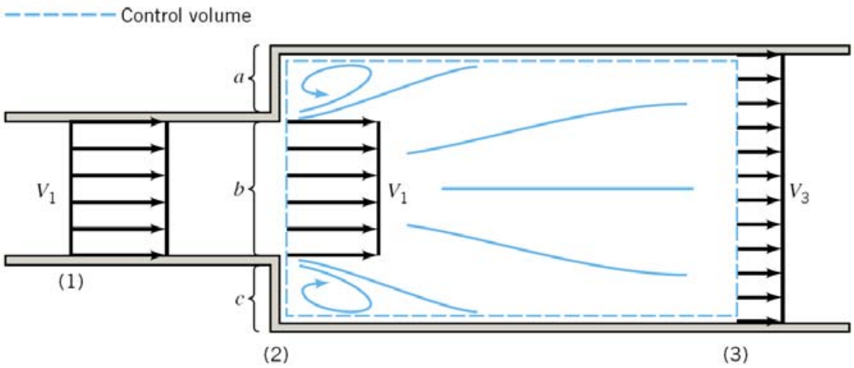
This checks with the previous horizontal result as is seen from the fact that a change in elevation of $\Delta z = \ell \sin\theta = (10\ \text{m})\sin(-13.34°) = -2.31$ m is equivalent to a pressure change of $\Delta p = \rho g\,\Delta z = (900\ \text{kg/m}^3)(9.81\ \text{m/s}^2)(2.31\ \text{m}) = 20{,}400$ N/m^2, which is equivalent to that needed for the horizontal pipe. For the horizontal pipe it is the work done by the pressure forces that overcomes the viscous dissipation. For the zero-pressure-drop pipe on the hill, it is the change in potential energy of the fluid "falling" down the hill that is converted to the energy lost by viscous dissipation. Note that if it is desired to increase the flowrate to $Q = 1.0 \times 10^{-4}\ \text{m}^3/\text{s}$ with $p_1 = p_2$, the value of θ given by Eq. 1 is $\sin\theta = -1.15$. Since the sine of an angle cannot be greater than 1, this flow would not be possible. The weight of the fluid would not be large enough to offset the viscous force generated for the flowrate desired. A larger diameter pipe would be needed.

(c) With $p_1 = p_2$ the length of the pipe, ℓ, does not appear in the flowrate equation (Eq. 1). This is a statement of the fact that for such cases the pressure is constant all along the pipe (provided the pipe lies on a hill of constant slope). This can be seen by substituting the values of Q and θ from case (b) into Eq. 8.12 and noting that $\Delta p = 0$ for any ℓ. For example, $\Delta p = p_1 - p_3 = 0$ if $\ell = x_3 - x_1 = 5$ m. Thus, $p_1 = p_2 = p_3$ so that

$$p_3 = 200 \text{ kPa} \qquad \textbf{(Ans)}$$

Note that if the fluid were gasoline ($\mu = 3.1 \times 10^{-4}\ \text{N} \cdot \text{s/m}^2$ and $\rho = 680\ \text{kg/m}^3$), the Reynolds number would be $\text{Re} = 2790$, the flow would probably not be laminar, and a use of Eqs. 8.9 and 8.12 would give incorrect results. Also note from Eq. 1 that the kinematic viscosity, $\nu = \mu/\rho$, is the important viscous parameter. This is a statement of the fact that with constant pressure along the pipe, it is the ratio of the viscous force ($\sim\mu$) to the weight force ($\sim\gamma = \rho g$) that determines the value of θ.

5. **Determine the head loss for a sudden expansion. Consider the control volume shown on the figure below and use conservation of mass and conservation of momentum.**



Mass Conservation

$\cancel{\rho_1} V_1 A_1 = \cancel{\rho_2} V_3 A_3 = \dot{m}$

density is constant

Momentum Conservation

$$\left.\begin{aligned} &(p_a A_a + p_b A_b + p_c A_c) - p_3 A_3 = \dot{M}_{out} - \dot{M}_{in} \\ &\dot{M}_{out} - \dot{M}_{in} = \dot{m}_{out} V_{out} - \dot{m}_{in} V_{in} = \dot{m}(V_3 - V_1) = \rho V_3 A_3 (V_3 - V_1) \\ &\text{Assume that } p_a = p_b = p_c \end{aligned}\right\} \Rightarrow$$

$$p_1 A_3 - p_3 A_3 = \rho V_3 A_3 (V_3 - V_1)$$

Energy Equation (Bernoulli's equation)

$$\frac{p_1}{\rho g}+\frac{V_1^2}{2g}=\frac{p_3}{\rho g}+\frac{V_3^2}{2g}+h_L$$

From momentum equation: $p_1 - p_3 = \rho V_3(V_3 - V_1)$

Substitute above in energy equation $\Rightarrow$

$$\cancel{\rho} V_3(V_3-V_1)+\frac{\cancel{\rho} V_1^2}{2}=\frac{\cancel{\rho} V_3^2}{2}+\cancel{\rho} g h_L$$

$$\left.\begin{array}{l} \text{Solve above for } gh_L = V_3^2 - V_3\ V_1 + \dfrac{V_1^2}{2} - \dfrac{V_3^2}{2} \\[2ex] \text{From mass conservation: } V_3 = \dfrac{V_1 A_1}{A_3} \end{array}\right\} \Rightarrow \text{Substitute } V_3$$

$$gh_L = \frac{1}{2}\left(\frac{V_1 A_1}{A_3}\right)^2 + \frac{V_1^2}{2} - \frac{V_1^2 A_1}{A_3} \quad \Rightarrow$$

$$\frac{gh_L}{V_1^2}=\frac{1}{2}\left(\frac{A_1}{A_3}\right)^2-\left(\frac{A_1}{A_3}\right)+\frac{1}{2}=\frac{1}{2}\left[\left(\frac{A_1}{A_3}\right)^2-2\left(\frac{A_1}{A_3}\right)+1\right]=\frac{1}{2}\left[1-\left(\frac{A_1}{A_3}\right)\right]^2$$

The loss coefficient $K_L = \dfrac{h_L}{\left(\dfrac{V_1^2}{2g}\right)} = \dfrac{2gh_L}{V_1^2} = \left[1-\left(\dfrac{A_1}{A_3}\right)\right]^2$

6. **Calculate the power supplied to the pump shown in Figure 3 if its efficiency is 76%. Methyl alcohol (**$\rho = 790\ \mathrm{kg/m^3},\ \mu = 5.6\times10^{-4}\ \mathrm{Pa\cdot s}$**) is flowing at the rate of** $54\ \mathrm{m^3/hr}$**. The suction line is a standard** $4-\mathrm{in}$ **steel pipe,** $15\ \mathrm{m}$ **long. The total length of** $2-\mathrm{in}$ **steel pipe in the discharge line is** $200\ \mathrm{m}$**. Assume that the entrance from reservoir 1 is through a squared-edged inlet and that the elbows are standard. The valve is a fully open globe valve. The roughness of the pipe is ϵ=**$0.045\ \mathrm{mm}$**.**

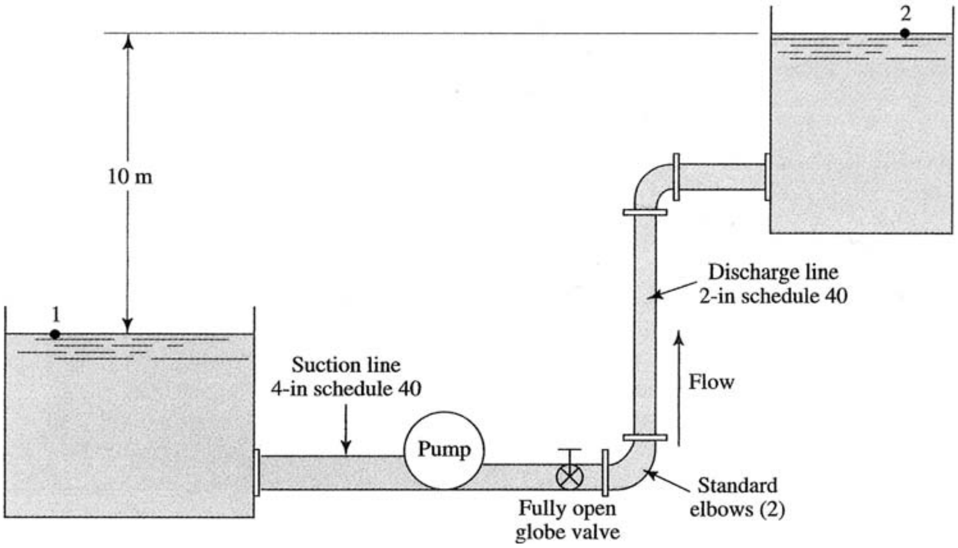


Figure 3: Pump/pipeline configuration

Consider a streamline joining the points 1 and 2. Applying the energy equation we obtain

$$p_1 + \frac{1}{2}\rho u_1^2 + \rho g z_1 + \frac{\dot{W}_{pump}}{Q} = p_2 + \frac{1}{2}\rho u_2^2 + \rho g z_2 + \rho g h_L.$$

p_1=p_2=p_{atm}. If we take as the datum the point 1 then $z_1 = 0$ and $z_2 = 10$ m.

If we further assume that $u_1 \approx 0$ and $u_2 \approx 0$ the energy equation simpilfies to

$$\dot{W}_{pump} = Q\left(\rho g z_2 + \rho g h_L\right).$$

Given:

$Q = 54 \text{ m}^3\text{/hr} = \dfrac{54}{3600} \text{ m}^3\text{/s=0.015 m}^3\text{/s}$

$D_{\text{suction}} = 4 \text{ in=0.1016 m}$

$\ell_{\text{suction}} = 15 \text{ m}$

$D_{\text{discharge}} = 2 \text{ in=0.0508 m}$

$\ell_{\text{discharge}} = 200 \text{ m}$

$\rho = 790 \text{ kg/m}^3$

$\mu = 5.6\times10^{-4} \text{ Pa}\cdot\text{s}$

$g = 9.81 \text{ m/s}^2$

$z_2 = 10 \text{ m}$

The only uknown in the equation for $\dot{W}_{pump}$ is

$$h_L = \underbrace{f\frac{\ell}{D}\frac{V^2}{2g}}_{\substack{\text{major losses}\\ \text{suction}}} + \underbrace{f\frac{\ell}{D}\frac{V^2}{2g}}_{\substack{\text{major losses}\\ \text{discharge}}} + \underbrace{K_L\frac{V^2}{2g}}_{\substack{\text{minor losses}\\ \text{pipe entrance}\\ K_L=0.5}} + \underbrace{K_L\frac{V^2}{2g}}_{\substack{\text{minor losses fully}\\ \text{open globe valve}\\ K_L=10}} + \underbrace{2K_L\frac{V^2}{2g}}_{\substack{\text{minor losses of}\\ \text{the 2 standard elbows}\\ K_L=0.3}} + \underbrace{K_L\frac{V^2}{2g}}_{\substack{\text{minor losses}\\ \text{pipe exit}\\ K_L=1}}$$

The loss coefficients can be obtained from a table, and the velocities from

$V_{\text{suction}} = Q/(\pi D^2_{\text{suction}}/4) = 4\times0.015/3.14/0.1016^2 = 1.85 \text{ m/s}$

$V_{\text{discharge}} = Q/(\pi D^2_{\text{discharge}}/4) = 4\times0.015/3.14/0.0508^2 = 7.4 \text{ m/s}$

To find the major losses we need to find the Reynolds number and the relative roughness

$$Re_{\text{suction}} = \frac{\rho V_{\text{suction}} D_{\text{suction}}}{\mu} = \frac{790\times1.85\times0.1016}{5.6\times10^{-4}} = 265000$$

$$\frac{\varepsilon}{D_{\text{suction}}} = \frac{0.045\times10^{-3}}{0.1016} = 0.00044$$

$f_{\text{suction}} = 0.019 \text{ (from Moody chart)}$

$$Re_{\text{discharge}} = \frac{\rho V_{\text{discharge}} D_{\text{discharge}}}{\mu} = \frac{790\times7.4\times0.0508}{5.6\times10^{-4}} = 530000$$

$$\frac{\varepsilon}{D_{\text{discharge}}} = \frac{0.045\times10^{-3}}{0.0508} = 0.000089$$

$f_{\text{discharge}} = 0.014 \text{ (from Moody chart)}$

Substitute all above information in the equation for h_L, calculate h_L and finally substitue in equation for $\dot{W}_{pump}$

7. **For the system shown in Figure 4, compute the power delivered by the pump to the water to pump $0.0031545\ \mathrm{m^3/s}$ of water at $15^o\ \mathrm{C}$ to the tank. The air in the tank is at $276\ \mathrm{kPa}$ gauge pressure. Consider the friction loss in the 225-ft-long discharge pipe, but neglect other losses. Then, redesign the system by using a larger pipe size to reduce the energy loss and reduce the power required to no more than $3729\ \mathrm{W}$. The roughness of the pipe is ϵ=1.5×10^{-4} and $1\ \mathrm{in}=0.0254\ \mathrm{m}$.**

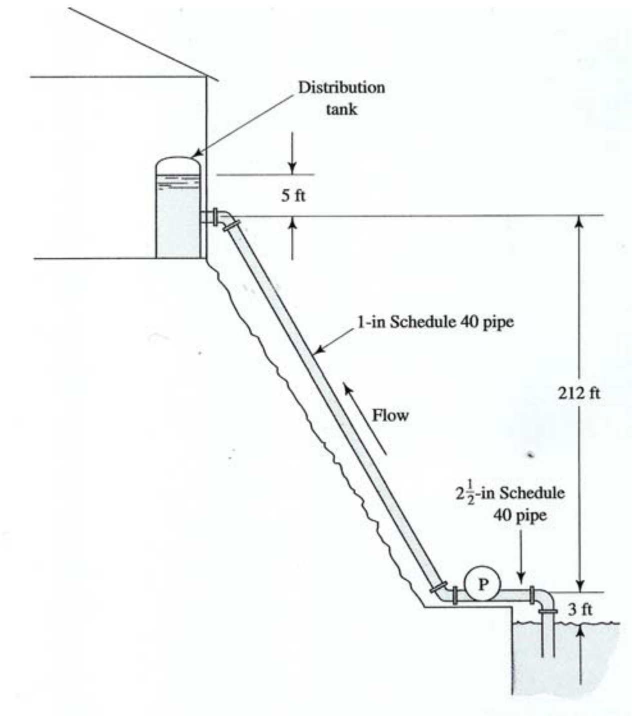


Figure 4: Pump/pipeline configuration

8. In the turbulent region the friction factor associated with pipe flow is approximated by the formula:

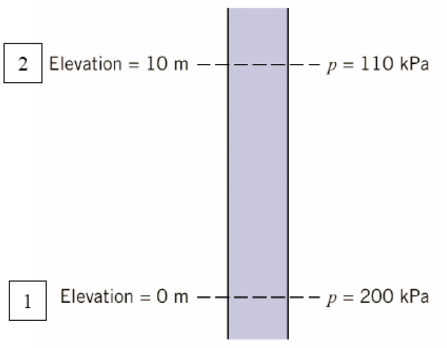
$$\sqrt{f} = \frac{0.5}{\log_{10}\left[\frac{\varepsilon}{3.7\,D} + \frac{5.74}{\mathrm{Re}^{0.9}}\right]}$$

Find an expression for the friction factor f for large Re number.

For large Reynolds number (Re) above expression simplifies to

$$\sqrt{f} = \frac{0.5}{\log_{10}\left[\frac{\varepsilon}{3.7\,D}\right]} \text{ because } \lim_{\mathrm{Re}\to\infty} \frac{5.74}{\mathrm{Re}^{0.9}} = 0.$$

Liquid with specific gravity $\gamma = \rho g = 10\ \mathrm{kN/m^3}$ is flowing in a vertical pipe. If the diameter of the pipe is $D = 15\ \mathrm{cm}$ and the viscosity of the fluid is $\mu = 3\times 10^{-3}\ \mathrm{N\cdot m/s^2}$ determine the direction of the flow and the mean velocity if the pipe relative roughness is $\varepsilon / D = 0.008$. The pressures shown are static pressures. Hint: Assume a high Reynolds number and verify.



Energy Equation (Bernoulli's equation)

$$\frac{p_1}{\rho g} + \frac{V_1^2}{2g} + z_1 = \frac{p_2}{\rho g} + \frac{V_2^2}{2g} + z_2 + h_L$$

where the losses are estimated using $h_L = f\,\frac{\ell}{D}\frac{u_m^2}{2g}$

and we have assumed that the flow is directed upwards.

Using mass conservation and assuming uniform flow
$V_1^2 = V_2^2 = u_m^2$.

So Bernoulli's equation simplies to

$$\frac{200000}{10000} + \frac{u_m^2}{2g} + 0 = \frac{110000}{10000} + \frac{u_m^2}{2g} + 10 + h_L \Rightarrow$$

$20 - 11 - 10 = h_L \Rightarrow h_L = -1$

Hence, our original assumption was wrong and the flow is directed downwards, i.e.

$$\frac{p_1}{\rho g} + \frac{V_1^2}{2g} + z_1 + h_L = \frac{p_2}{\rho g} + \frac{V_2^2}{2g} + z_2$$

$h_L = 1$

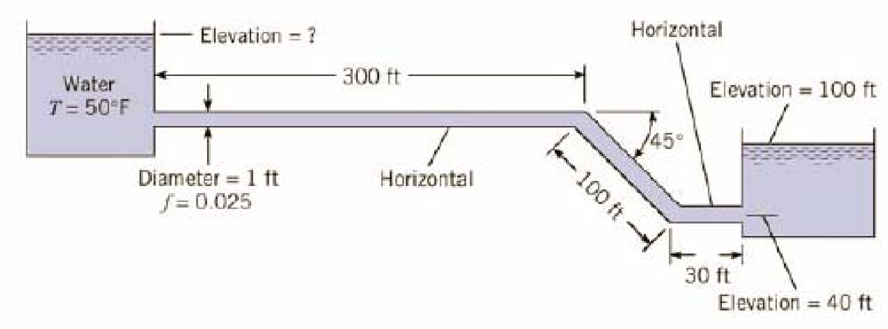
If we assume that the flow has a high Reynolds number

then $f = \dfrac{0.25}{\left(\log_{10}\left[\dfrac{\varepsilon}{3.7\,D}\right]\right)^2} = \dfrac{0.25}{\left(\log_{10}\left[\dfrac{0.008}{3.7}\right]\right)^2} = 0.0352$

$$h_L = 1 = 0.0352 \frac{10}{0.15} \frac{u_m^2}{2g} \Rightarrow 0.12\, u_m^2 = 1 \Rightarrow u_m = 2.89 \text{ m/s}$$

Verify Reynolds number $\text{Re} = \dfrac{\rho u D}{\mu} = \dfrac{1019 \cdot 2.89 \cdot 0.15}{3 \cdot 10^{-3}} = 144500$

9. Estimate the elevation required in the upper reservoir to produce a water discharge of 10 cfs in the system. What is the minimum pressure in the pipeline and what is the pressure there?



$$\frac{V_1^2}{2g}+\frac{p_1}{\gamma}+z_1-\sum h_L=\alpha_2\frac{V_2^2}{2g}+\frac{p_2}{\gamma}+z_2$$

$$0+0+z_1-\sum h_L=0+0+z_2$$

$$\sum h_L=\left(K_e+2K_b+K_E+f\frac{L}{D}\right)\frac{V^2}{2g}$$

$K_e=0.5;\ K_b=0.4\,\text{(assumed)};\ K_E=1.0;\ f\frac{L}{D}=0.025*\frac{430}{1}=10.75$

$$V=\frac{Q}{A}=\frac{10}{\pi/4*1^2}=12.73\ ft/s$$

$$z_1=100+(0.5+2*0.4+1.0+10.75)\frac{12.73^2}{2*32.2}=133\ ft$$

$$\frac{V_1^2}{2g}+\frac{p_1}{\gamma}+z_1-\sum h_L=\alpha_b\frac{V_b^2}{2g}+\frac{p_b}{\gamma}+z_b$$

$$0+0+z_1-\sum h_L=1*\frac{V_b^2}{2g}+\frac{p_b}{\gamma}+z_b$$

$$\frac{p_b}{\gamma}=z_1-z_b-\frac{V_b^2}{2g}-\left(K_e+K_b+f\frac{L}{D}\right)\frac{V^2}{2g}$$

$$=133-110.7-\left(1.0+0.5+0.4+0.025\frac{300}{1}\right)\frac{12.73^2}{2*32.2}$$

$$=-1.35\ ft$$

$$p_b=62.4*(-1.53)=-0.59\ psig$$

$$\text{Re}=\frac{VD}{\nu}=\frac{12.73*1}{1.14*10^{-5}}=9*10^5$$

10. Water flows from a reservoir through a pipe 150mm diameter and 180m long to a point below the surface of the reservoir where it branches into two pipes, each 100mm in diameter (see Figure 2). One of the pipes is 48m long discharging to atmosphere at a point below reservoir level and the other 60m long discharging to atmosphere 24m below reservoir level. Assuming that $f = 0.032$ calculate the discharge from each pipe, neglecting all loses other than friction.

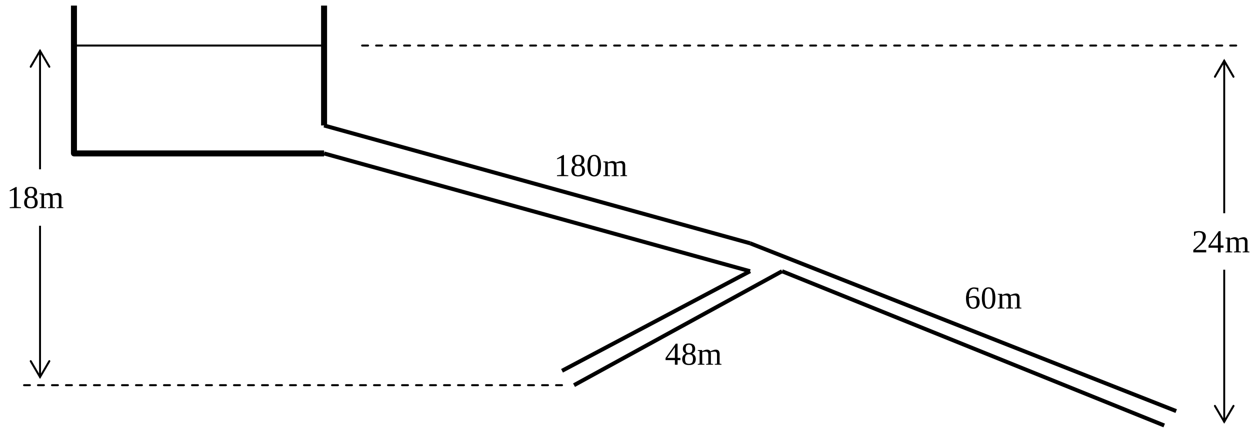
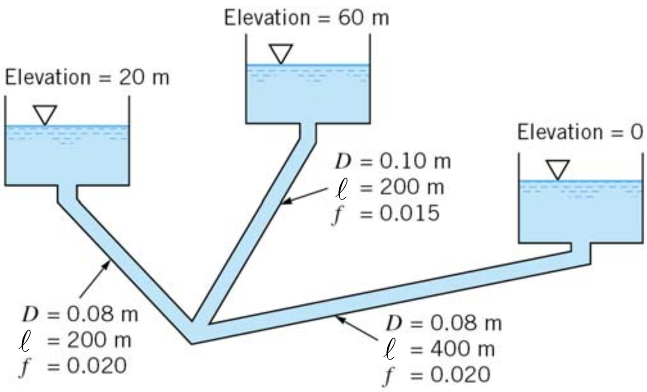


Figure 2: Reservoir pipeline configuration

11. The three water-filled tanks shown in the figure (Figure P8.102 in textbook) are connected by pipes as indicated in the figure. If minor losses are neglected determine the flowrate in each pipe.



8.63

8.63 The three water-filled tanks shown in Fig. P8.63 are connected by pipes as indicated. If minor losses are neglected, determine the flowrate in each pipe.

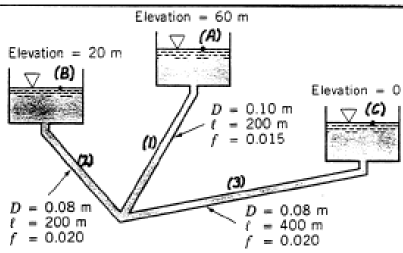


FIGURE P8.63

Assume the fluid flows from A to B and A to C. Thus, $Q_1 = Q_2 + Q_3$

or $\frac{\pi}{4}(0.1m)^2 V_1 = \frac{\pi}{4}(0.08m)^2 V_2 + \frac{\pi}{4}(0.08m)^2 V_3$

Thus, $V_1 = 0.64 V_2 + 0.64 V_3$ (1)

For fluid flowing from A to B with $p_A = p_B = 0$ and $V_A = V_B = 0$,

$z_A = z_B + f_1 \frac{\ell_1}{D_1}\frac{V_1^2}{2g} + f_2 \frac{\ell_2}{D_2}\frac{V_2^2}{2g}$

or

$60m - 20m = (0.015)\left(\frac{200m}{0.1m}\right)\frac{V_1^2}{2(9.81\frac{m}{s^2})} + (0.020)\left(\frac{200m}{0.08m}\right)\frac{V_2^2}{2(9.81\frac{m}{s^2})}$

Hence,

$40 = 1.529 V_1^2 + 2.55 V_2^2$ (2)

Similarly, for fluid flowing from A to C with $p_A = p_C = 0$ and $V_A = V_C = 0$,

$z_A = z_C + f_1 \frac{\ell_1}{D_1}\frac{V_1^2}{2g} + f_3 \frac{\ell_3}{D_3}\frac{V_3^2}{2g}$

or

$60m = (0.015)\left(\frac{200m}{0.1m}\right)\frac{V_1^2}{2(9.81\frac{m}{s^2})} + (0.020)\left(\frac{400m}{0.08m}\right)\frac{V_3^2}{2(9.81\frac{m}{s^2})}$

Hence,

$60 = 1.529 V_1^2 + 5.10 V_3^2$ (3)

Solve Eqs. (1), (2), and (3) for V_1, V_2, and V_3. From Eqs. (1) and (3):

$60 = 1.529(0.64)^2(V_2 + V_3)^2 + 5.10 V_3^2$, or $95.8 = (V_2 + V_3)^2 + 8.14 V_3^2$ (4)

Subtract Eq. (2) from Eq. (3):

$60 - 40 = 5.10 V_3^2 + 2.55 V_2^2$ or $V_2 = \sqrt{2V_3^2 - 7.84}$ (5)

Thus, from Eqs. (4) and (5): $8.14 V_3^2 + \left(\sqrt{2V_3^2 - 7.84} + V_3\right)^2 - 95.8 = 0$

This can be simplified to

$2V_3\sqrt{2V_3^2 - 7.84} = 103.6 - 11.14 V_3^2$ Square both sides and (6)

rearrange to give $V_3^4 - 19.63 V_3^2 + 92.5 = 0$ which can be solved by the quadratic formula to give

$V_3^2 = \frac{19.63 \pm \sqrt{19.63^2 - 4(92.5)}}{2} = 11.77$ or 7.86 Thus $V_3 = 3.43 \frac{m}{s}$

or $V_3 = 2.80 \frac{m}{s}$

Note: The value $V_3 = 3.43 \frac{m}{s}$ is not a solution of the original equations, Eqs. (1), (2), and (3). With this value the right hand side of Eq. (6) is negative (i.e. $103.6 - 11.14 V_3^2 = 103.6 - 11.14(3.43)^2 = -24.5$). As seen from the left hand side of Eq. (6), this cannot be. This extra root was introduced by squaring Eq. (6).

Thus, $Q_3 = A_3 V_3 = \frac{\pi}{4}(0.08m)^2 (2.80 \frac{m}{s}) = \underline{\underline{0.0141 \frac{m^3}{s}}}$

Also, from Eq. (3):

$$60 = 1.529 V_1^2 + 5.10(2.80)^2 \quad \text{or} \quad V_1 = 3.62 \frac{m}{s}$$

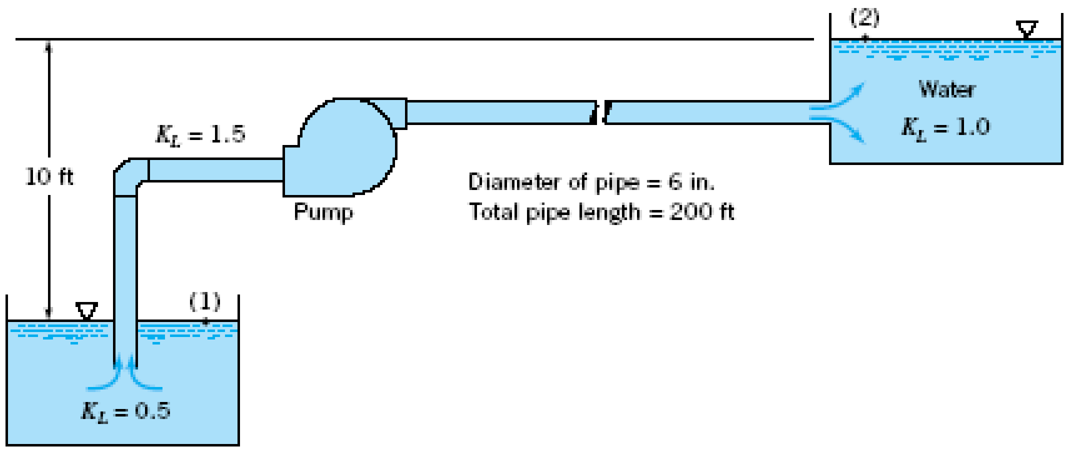
or $Q_1 = A_1 V_1 = \frac{\pi}{4}(0.10m)^2 (3.62 \frac{m}{s}) = \underline{\underline{0.0284 \frac{m^3}{s}}}$

and from Eq. (1):

$$3.62 = 0.64 V_2 + 0.64(2.80) \quad \text{or} \quad V_2 = 2.86 \frac{m}{s}$$

or $Q_2 = A_2 V_2 = \frac{\pi}{4}(0.08m)^2 (2.86 \frac{m}{s}) = \underline{\underline{0.0143 \frac{m^3}{s}}}$

12. **Water is to be pumped from one large, open tank to a second large, open tank as shown in the figure. The pipe diameter throughout is $15\ \text{cm}$ and the total length of the pipe between the pipe entrance and exit is $61\ \text{m}$. Minor loss coefficients for the entrance, exit, and the elbow are shown on the figure, and the friction factor for the pipe can be assumed constant and equal to 0.02. A certain centrifugal pump having the performance characteristics shown in the figure is suggested as a good pump for this flow system. With this pump, what would be the flowrate between the tanks? Do you think this pump would be a good choice?**



(a)

SOLUTION

Application of the energy equation between the two free surfaces, points (1) and (2) as indicated, gives

$$\frac{p_1}{\gamma} + \frac{V_1^2}{2g} + z_1 + h_p = \frac{p_2}{\gamma} + \frac{V_2^2}{2g} + z_2 + f\frac{\ell}{D}\frac{V^2}{2g} + \sum K_L \frac{V^2}{2g} \quad \textbf{(1)}$$

Thus, with $p_1 = p_2 = 0$, $V_1 = V_2 = 0$, $z_2 - z_1 = 10$ ft, $f = 0.02$, $D = 6/12$ ft, and $\ell =$ 200 ft, Eq. (1) becomes

$$h_p = 10 + \left[0.02\frac{(200\text{ ft})}{(6/12\text{ ft})} + (0.5 + 1.5 + 1.0)\right]\frac{V^2}{2(32.2\text{ ft/s}^2)} \quad \textbf{(2)}$$

where the given minor loss coefficients have been used. Since

$$V = \frac{Q}{A} = \frac{Q(\text{ft}^3/\text{s})}{(\pi/4)(6/12\text{ ft})^2}$$

Eq. 2 can be expressec

$$h_p = 10 + 4.43\,Q^2 \quad \textbf{(3)}$$

where Q is in ft^3/s, or with Q in gal/min,

$$h_p = 10 + 2.20 \times 10^{-5} Q^2 \quad \textbf{(4)}$$

Equation 3 or 4 represents the system equation for this particular flow system and reveals how much actual head the fluid will need to gain from the pump to maintain a certain flowrate. The performance data shown in Fig. E12.4*b* indicate the actual head the fluid will gain from this particular pump when it operates at a certain flowrate. Thus, when Eq. 4 is plotted on the same graph with the performance data, the intersection of the two curves represents the operating point for the pump and the system. This combination is shown in Fig. E12.4*c* with the intersection (as obtained graphically) occurring at

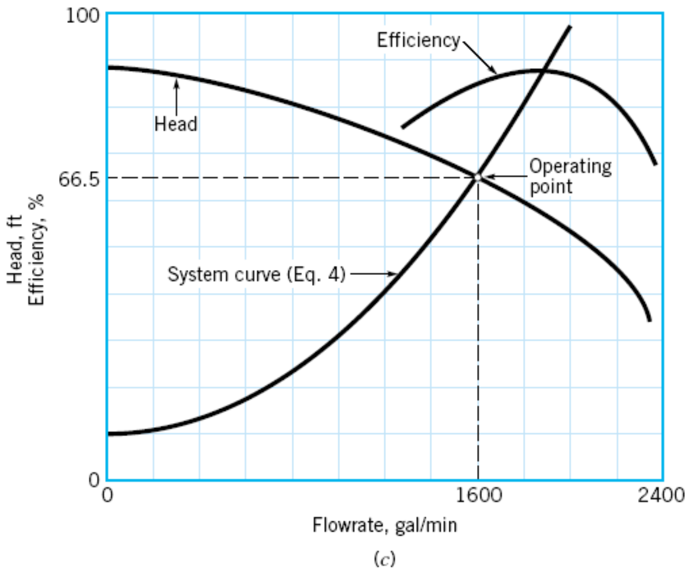
$$Q = 1600\text{ gal/min} \quad \textbf{(Ans)}$$

with the corresponding actual head gained equal to 66.5 ft.

Another concern is whether or not the pump is operating efficiently at the operating point. As can be seen from Fig. E12.4*c*, although this is not peak efficiency, which is about 86%, it is close (about 84%). Thus, this pump would be a satisfactory choice, assuming the 1600 gal/min flowrate is at or near the desired flowrate.

The amount of pump head needed at the pump shaft is

$$\frac{66.5 \text{ ft}}{0.84} = 79.2 \text{ ft}$$



■ **FIGURE E12.4** **(*Continued*)**

The power needed to drive the pump is

$$\dot{W}_{\text{shaft}} = \frac{\gamma Q h_a}{\eta}$$

$$= \frac{(62.4 \text{ lb/ft}^3)[(1600 \text{ gal/min})/(7.48 \text{ gal/ft}^3)(60 \text{ s/min})](66.5 \text{ ft})}{0.84}$$

$$= 17{,}600 \text{ ft} \cdot \text{lb/s} = 32.0 \text{ hp}$$