FREDERICK UNIVERSITY CYPRUS

Department of Mechanical Engineering

AMEE401 / AUTO400
Aerodynamics

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HOMEWORK ASSIGNMENT #3 SOLUTION

QUESTION 1

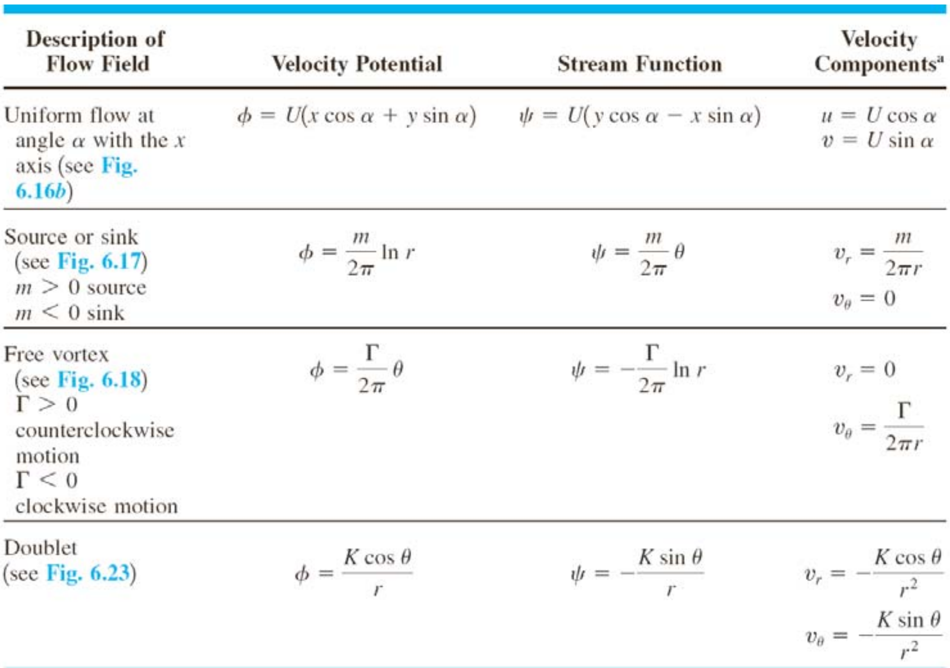
For incompressible, two-dimensional, inviscid flow in cartesian coordinates the mass conservation equation states that:

$$\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} = 0 .$$

In cylindrical coordinates the equivalent equation is:

$$\frac{\partial (r u_r)}{\partial r} + \frac{\partial u_\theta}{\partial \theta} = 0$$

Show that the following flow fields satisfy above expressions:



Description of Flow Field	Velocity Potential	Stream Function	Velocity Components[a]
Uniform flow at angle α with the x axis (see Fig. 6.16*b*)	$\phi = U(x\cos\alpha + y\sin\alpha)$	$\psi = U(y\cos\alpha - x\sin\alpha)$	$u = U\cos\alpha$ $v = U\sin\alpha$
Source or sink (see Fig. 6.17) $m > 0$ source $m < 0$ sink	$\phi = \frac{m}{2\pi}\ln r$	$\psi = \frac{m}{2\pi}\theta$	$v_r = \frac{m}{2\pi r}$ $v_\theta = 0$
Free vortex (see Fig. 6.18) $\Gamma > 0$ counterclockwise motion $\Gamma < 0$ clockwise motion	$\phi = \frac{\Gamma}{2\pi}\theta$	$\psi = -\frac{\Gamma}{2\pi}\ln r$	$v_r = 0$ $v_\theta = \frac{\Gamma}{2\pi r}$
Doublet (see Fig. 6.23)	$\phi = \frac{K\cos\theta}{r}$	$\psi = -\frac{K\sin\theta}{r}$	$v_r = -\frac{K\cos\theta}{r^2}$ $v_\theta = -\frac{K\sin\theta}{r^2}$

For **uniform flow** the equations are in cartesian coordinates, hence we have to use $\frac{\partial u_x}{\partial x}+\frac{\partial u_y}{\partial y}=0$.

Because u_x and u_y are simply cosntants $\frac{\partial u_x}{\partial x}=0$ and $\frac{\partial u_y}{\partial y}=0$, hence above equation is identically satisfied.

For the other flows the equation in cylindrical-polar coordinates has to be used, i.e. $\frac{\partial (ru_r)}{\partial r}+\frac{\partial u_\theta}{\partial \theta}=0$.

Source - sink

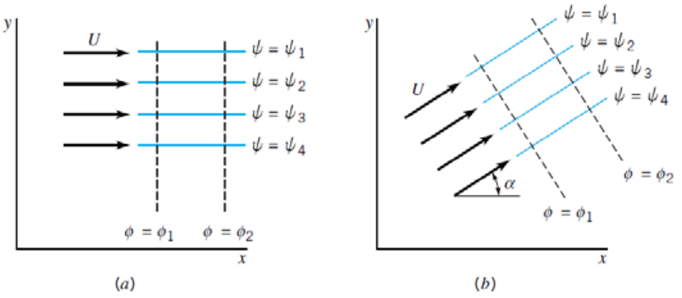
$$\frac{\partial (ru_r)}{\partial r}=\frac{\partial\left(r\frac{\Gamma}{2\pi r}\right)}{\partial r}=\frac{\partial\left(\frac{\Gamma}{2\pi}\right)}{\partial r}=0,\quad \frac{\partial u_\theta}{\partial \theta}=0$$

Free vortex

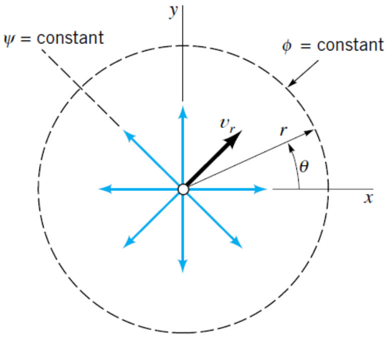
$$\frac{\partial (ru_r)}{\partial r}=0,\quad \frac{\partial\left(\frac{\Gamma}{2\pi r}\right)}{\partial \theta}=0$$

Doublet

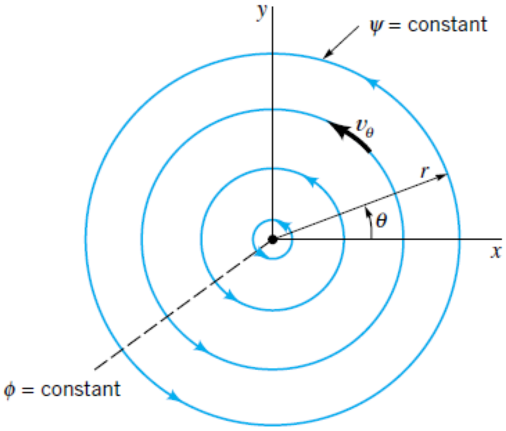
$$\left.\begin{aligned}\frac{\partial (ru_r)}{\partial r}&=\frac{\partial\left(r\frac{-\mathrm{K}\cos\theta}{r^2}\right)}{\partial r}=-\mathrm{K}\cos\theta\frac{\partial\left(\frac{1}{r}\right)}{\partial r}=\frac{\mathrm{K}\cos\theta}{r^2}\\ \frac{\partial u_\theta}{\partial \theta}&=\frac{\partial\frac{-\mathrm{K}\sin\theta}{r^2}}{\partial\theta}=\frac{-\mathrm{K}}{r^2}\frac{\partial\sin\theta}{\partial\theta}=\frac{-\mathrm{K}}{r^2}\cos\theta\end{aligned}\right\}\Rightarrow\frac{\partial (ru_r)}{\partial r}+\frac{\partial u_\theta}{\partial \theta}=\frac{\mathrm{K}\cos\theta}{r^2}+\frac{-\mathrm{K}}{r^2}\cos\theta=0$$



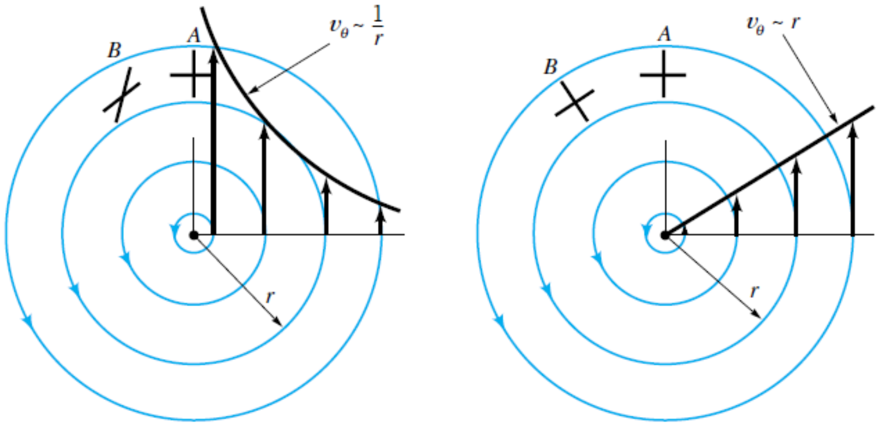
■ **FIGURE 6.16** **Uniform flow: (*a*) in the *x* direction; (*b*) in an arbitrary direction, *α*.**



■ **FIGURE 6.17** **The streamline pattern for a source.**



■ FIGURE 6.18 The streamline pattern for a vortex.



■ FIGURE 6.19 Motion of fluid element from *A* to *B*: (*a*) for irrotational (free) vortex; (*b*) for rotational (forced) vortex.

QUESTION 2

The two-dimensional flow of a nonviscous, incompressible fluid in the vicinity of a corner (see figure) is described by the stream function $\psi = 2r^2 \sin 2\theta$, where ψ has units of $\mathrm{m^2/s}$ when r is in meters. Determine:

a. The radial and tangential components of the velocity

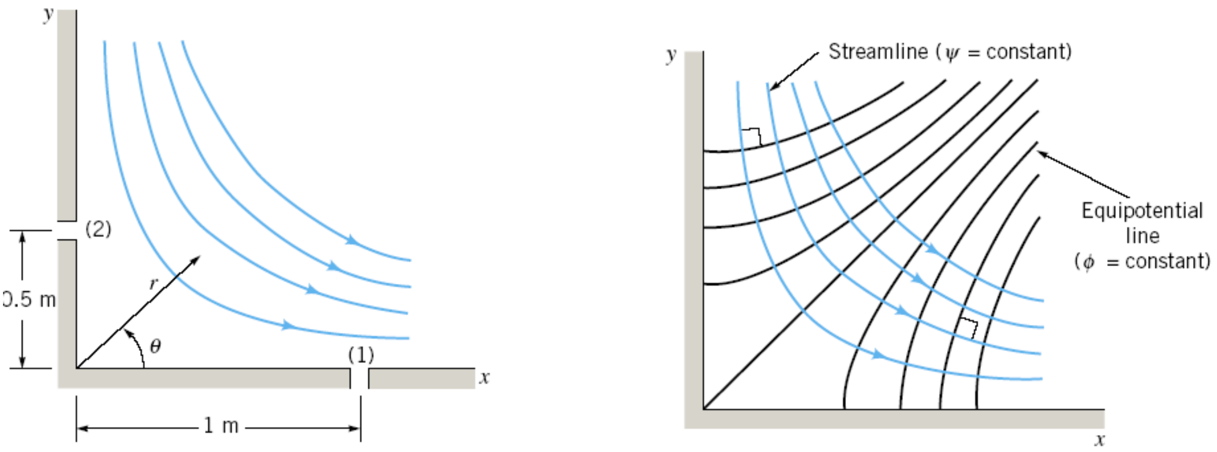
b. The corresponding velocity potential.

c. If the pressure at point (1) on the wall is 30 kPa, what is the pressure at point (2)?

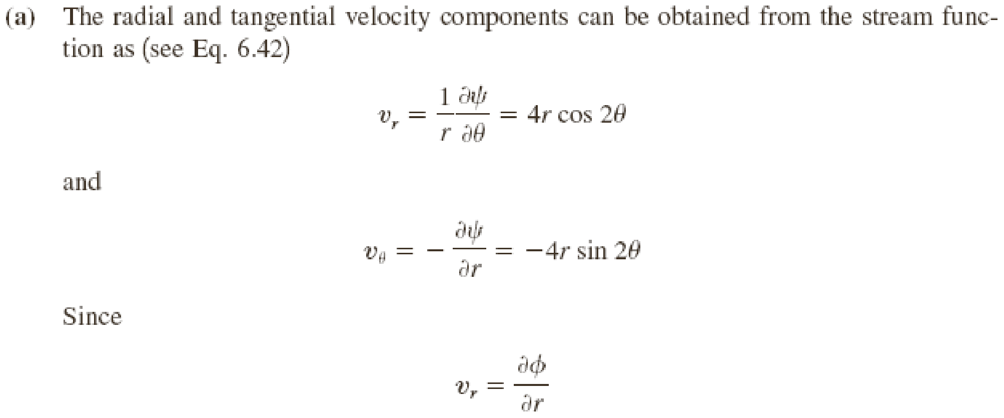
d. Express the velocity potential in Cartesian coordinates and comment on the character of the equipotential lines.

e. Determine the mass flow rate across the points $(x, y) = (2, 2)$ and $(x, y) = (1,1)$.

Assume the fluid density is $\rho = 10^3\ \mathrm{kg/m^3}$ and the x–y plane is horizontal—that is, there is no difference in elevation between points (1) and (2).



(*a*) (*b*)



(a) The radial and tangential velocity components can be obtained from the stream function as (see Eq. 6.42)

$$v_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} = 4r \cos 2\theta$$

and

$$v_\theta = -\frac{\partial \psi}{\partial r} = -4r \sin 2\theta$$

Since

$$v_r = \frac{\partial \phi}{\partial r}$$

it follows that

$$\frac{\partial \phi}{\partial r} = 4r \cos 2\theta$$

and therefore by integration

$$\phi = 2r^2 \cos 2\theta + f_1(\theta) \qquad \textbf{(1)}$$

where $f_1(\theta)$ is an arbitrary function of θ. Similarly

$$v_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta} = -4r \sin 2\theta$$

and integration yields

$$\phi = 2r^2 \cos 2\theta + f_2(r) \qquad \textbf{(2)}$$

where $f_2(r)$ is an arbitrary function of r. To satisfy both Eqs. 1 and 2, the velocity potential must have the form

$$\phi = 2r^2 \cos 2\theta + C \qquad \textbf{(Ans)}$$

where C is an arbitrary constant. As is the case for stream functions, the specific value of C is not important, and it is customary to let $C = 0$ so that the velocity potential for this corner flow is

$$\phi = 2r^2 \cos 2\theta \qquad \textbf{(Ans)}$$

In the statement of this problem it was implied by the wording "if possible" that we might not be able to find a corresponding velocity potential. The reason for this concern is that we can always define a stream function for two-dimensional flow, but the flow must be *irrotational* if there is a corresponding velocity potential. Thus, the fact that we were able to determine a velocity potential means that the flow is irrotational. Several streamlines and lines of constant ϕ are plotted in Fig. E6.4*b*. These two sets of lines are *orthogonal*. The reason why streamlines and lines of constant ϕ are always orthogonal is explained in Section 6.5.

(b) Since we have an irrotational flow of a nonviscous, incompressible fluid, the Bernoulli equation can be applied between any two points. Thus, between points (1) and (2) with no elevation change

$$\frac{p_1}{\gamma} + \frac{V_1^2}{2g} = \frac{p_2}{\gamma} + \frac{V_2^2}{2g}$$

or

$$p_2 = p_1 + \frac{\rho}{2}(V_1^2 - V_2^2) \qquad \textbf{(3)}$$

Since

$$V^2 = v_r^2 + v_\theta^2$$

it follows that for any point within the flow field

$$\begin{aligned} V^2 &= (4r \cos 2\theta)^2 + (-4r \sin 2\theta)^2 \\ &= 16r^2(\cos^2 2\theta + \sin^2 2\theta) \\ &= 16r^2 \end{aligned}$$

This result indicates that the square of the velocity at any point depends only on the radial distance, r, to the point. Note that the constant, 16, has units of s^{-2}. Thus,

$$V_1^2 = (16\ \text{s}^{-2})(1\ \text{m})^2 = 16\ \text{m}^2/\text{s}^2$$

and

$$V_2^2 = (16\ \text{s}^{-2})(0.5\ \text{m})^2 = 4\ \text{m}^2/\text{s}^2$$

Substitution of these velocities into Eq. 3 gives

$$p_2 = 30 \times 10^3\ \text{N/m}^2 + \frac{10^3\text{kg/m}^3}{2}(16\ \text{m}^2/\text{s}^2 - 4\ \text{m}^2/\text{s}^2) = 36\ \text{kPa} \qquad \textbf{(Ans)}$$

The stream function used in this example could also be expressed in Cartesian coordinates as

$$\psi = 2r^2 \sin 2\theta = 4r^2 \sin\theta \cos\theta$$

or

$$\psi = 4xy$$

since $x = r\cos\theta$ and $y = r\sin\theta$. However, in the cylindrical polar form the results can

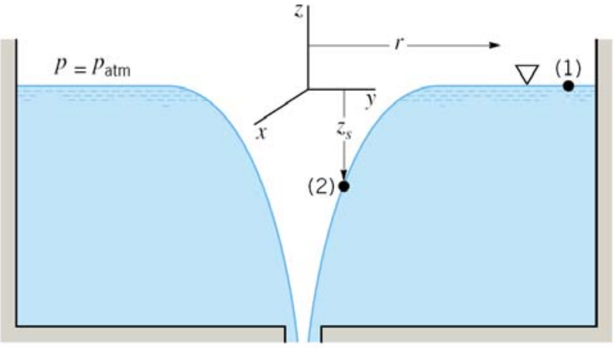
QUESTION 3

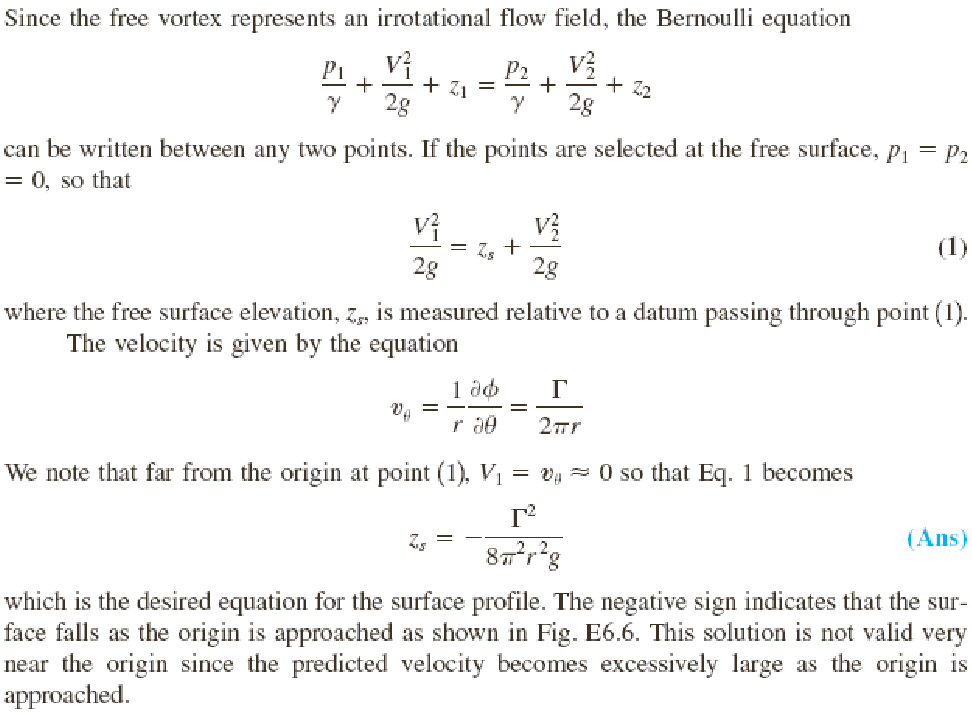
A liquid drains from a large tank through a small opening as illustrated in the figure. A vortex forms whose velocity distribution away from the tank opening can be approximated as that of a free vortex having a velocity potential $\phi = \frac{\Gamma}{2\pi}\theta$.

Determine:

a. The streamfunction and draw the streamline pattern along with the velocity profile.

b. an expression relating the surface shape to the strength of the vortex as specified by the circulation Γ.





Since the free vortex represents an irrotational flow field, the Bernoulli equation

$$\frac{p_1}{\gamma} + \frac{V_1^2}{2g} + z_1 = \frac{p_2}{\gamma} + \frac{V_2^2}{2g} + z_2$$

can be written between any two points. If the points are selected at the free surface, $p_1 = p_2 = 0$, so that

$$\frac{V_1^2}{2g} = z_s + \frac{V_2^2}{2g} \qquad (1)$$

where the free surface elevation, z_s, is measured relative to a datum passing through point (1).

The velocity is given by the equation

$$v_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta} = \frac{\Gamma}{2\pi r}$$

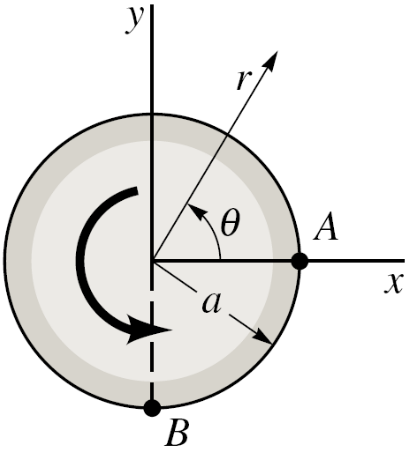
We note that far from the origin at point (1), $V_1 = v_\theta \approx 0$ so that Eq. 1 becomes

$$z_s = -\frac{\Gamma^2}{8\pi^2 r^2 g} \qquad \text{(Ans)}$$

which is the desired equation for the surface profile. The negative sign indicates that the surface falls as the origin is approached as shown in Fig. E6.6. This solution is not valid very near the origin since the predicted velocity becomes excessively large as the origin is approached.

QUESTION 4

The velocity potential for a cylinder (see Figure) rotating in a uniform stream of fluid is $\phi = Ur\left(1+\frac{a^2}{r^2}\right)\cos\theta + \frac{\Gamma}{2\pi}\theta$ where Γ is the circulation. Define a nondimensional rotation rate by $\omega^* = \omega D/2U$ where D is the diameter, ω is the rotation rate and U is the uniform stream velocity.

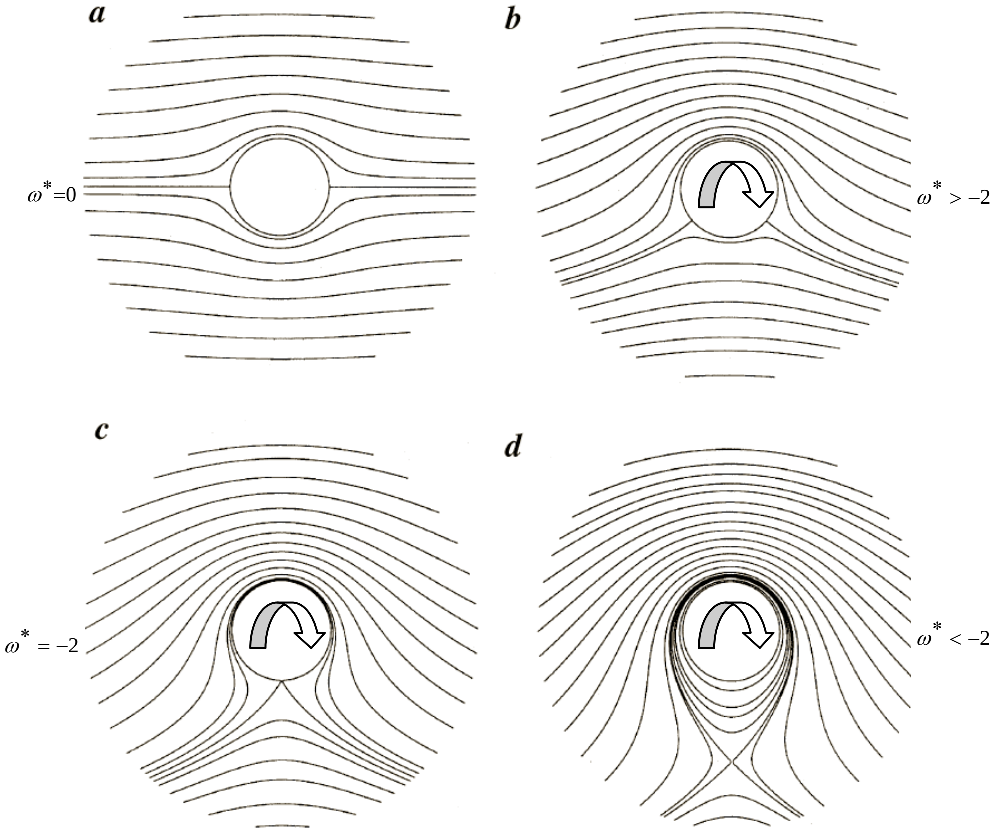


Hence show that the critical value of ω^* such that the stagnation point is at point B is $\omega^* = -2$.

The rotation of the sphere is simulated with a free vortex, hence the tangential velocity on the sphere is given by the tangentail velocity of the free vortex at $r = a,$ i.e. $u_\theta = \frac{\Gamma}{2\pi a}$. For a rotating disk the tangential velocity is related to the angular velocity through $u_\theta = \omega r,$ hence $u_\theta = \omega a$. Combining above two equations we obtain $\boxed{\Gamma = 2\pi a^2\omega}$.

The tangential velocity of the flow can be obtain using the velocity potential through $u_\theta = \frac{1}{r}\frac{\partial\phi}{\partial\theta} = -U\left(1+\frac{a^2}{r^2}\right)\sin\theta + \frac{\Gamma}{2\pi r}$. Hence, on the surface of the cylinder the velocity is $u_\theta = -2U\sin\theta + \frac{\Gamma}{2\pi a}$ and the stagnation points are the points where the velocity is zero and they are located at the angle $\sin\theta = \frac{\Gamma}{4\pi aU}$. If the stagnation point is located at point B the $\theta = -\pi/2$ hence $\sin\theta = -1$ hence $\boxed{\Gamma = -4\pi aU}$ which implies a a clockwise rotating vortex. Combining the two equaitons we obtain $-4\pi aU = 2\pi a^2\omega$ which can be simplfied to $\omega = -\frac{2U}{a} = -\frac{4U}{D}$. Hence $\frac{\omega \mathrm{D}}{2U} = -2 = \omega^*$

Sketch the streamlines for different values of ω^*.

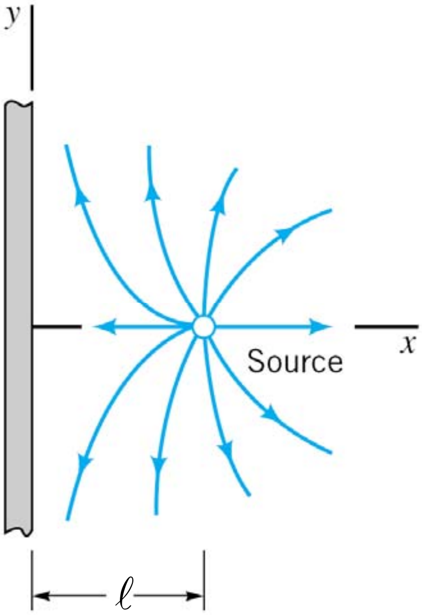


QUESTION 5

A source of strength m is located a distance ℓ from a vertical solid wall as shown in the Figure. The velocity potential for this incompressible, irrotational flow is given by

$$\phi = \frac{m}{4\pi}\left\{\ln\left[(x-\ell)^2 + y^2\right] + \ln\left[(x+\ell)^2 + y^2\right]\right\}$$

(a) Show that there is no flow through the wall. **(b)** Determine the velocity distribution along the wall. **(c)** Determine the pressure distribution along the wall, assuming $p = p_0$ far from the source. Neglect the effect of the fluid weight on the pressure.



The velocity in the x-direction can be obtained through:

$$u = \frac{\partial\phi}{\partial x} = \frac{m}{4\pi}\left\{\frac{2(x-\ell)}{(x-\ell)^2 + y^2} + \frac{2(x+\ell)}{(x+\ell)^2 + y^2}\right\}$$

If there is no flow through the wall then $u = 0$ at the location of the wall.

Note that the vertical velocity would not be zero as we have assumed

that the flow is inviscid. The wall is located at $x = 0$. So

$$u(x=0) = \frac{m}{4\pi}\left\{\frac{-2\ell}{\ell^2 + y^2} + \frac{2\ell}{\ell^2 + y^2}\right\} = 0$$

The vertical velocity along the wall v is obtained through:

$$v(x=0) = \frac{\partial\phi}{\partial y} = \frac{m}{4\pi}\left\{\frac{2y}{(x-\ell)^2 + y^2} + \frac{2y}{(x+\ell)^2 + y^2}\right\} = \frac{m}{4\pi}\left\{\frac{2y}{(\ell)^2 + y^2} + \frac{2y}{(\ell)^2 + y^2}\right\} = \frac{my}{4\pi\left(\ell^2 + y^2\right)}$$

The pressure along the wall can be obtained through Bernoulli's equation

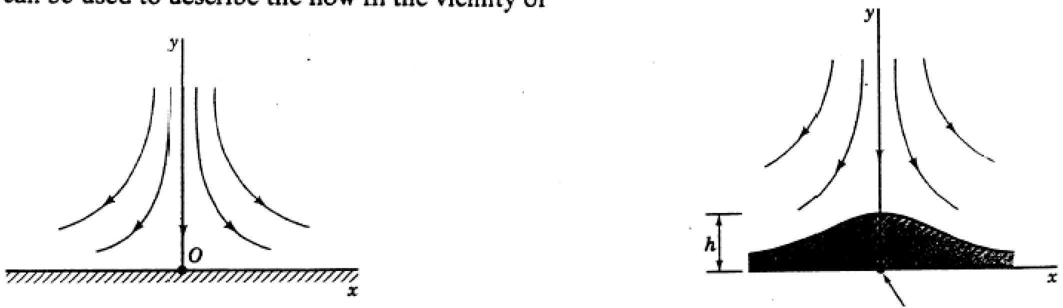
$$p + \frac{1}{2}\rho|V|^2 = p_0 \Rightarrow p = p_0 - \frac{1}{2}\rho(u^2 + v^2) = p_0 - \frac{\rho m^2 y^2}{2\pi^2\left(\ell^2 + y^2\right)^2}$$

QUESTION 6

Potential flow against a flat plate (Fig. P6.28 *a*) can be described with the stream function

$$\psi = Axy$$

where A is a constant. This type of flow is commonly called a "stagnation point" flow since it can be used to describe the flow in the vicinity of the stagnation point at O. By adding a source of strength, m, at O, stagnation point flow against a flat plate with a "bump" is obtained as illustrated in Fig. P6.28*b*. Determine the relationship between the bump height, h, the constant, A; and the source strength, m.



(*a*) (*b*)

FIGURE P6.28

$$\psi = Axy + \frac{m}{2\pi}\theta = \frac{A}{2}r^2 \sin 2\theta + \frac{m}{2\pi}\theta$$

For the bump the stagnation point will occur at $x = 0$, $y = h$ $(\theta = \frac{\pi}{2}, r = h)$. For the given stream function,

$$v_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} = Ar \cos 2\theta + \frac{m}{2\pi r} \qquad (1)$$

and

$$v_\theta = -\frac{\partial \psi}{\partial r} = Ar \sin 2\theta$$

The point, $\theta = \frac{\pi}{2}$, $r = h$, will be a stagnation point if $v_r = 0$ since $v_\theta = 0$ at this point. Thus, from Eq. (1)

$$0 = Ah \cos \pi + \frac{m}{2\pi h}$$

or

$$Ah = \frac{m}{2\pi h}$$

and therefore

$$\underline{\underline{h^2 = \frac{m}{2\pi A}}}$$