

Critical insulation thickness of a rectangular slab embedded with a periodic array of isothermal strips

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Abstract

We address the problem of two-dimensional heat conduction in a solid slab embedded with a periodic array of isothermal strips. The surfaces of the slab are subjected to a convective heat transfer boundary condition with a uniform heat transfer coefficient. Similar to the concept of critical insulation radius, associated with cylindrical and spherical configurations, we show that there exists a critical insulation thickness, associated with the slab, such that the total thermal resistance attains a minimum, i.e. a maximum heat transfer rate can be achieved. This result, which is not observed in one-dimensional heat conduction in a plane wall, is a consequence of the non-trivial coupling between conduction and convection that results in a 2D temperature distribution in the slab, and a non-uniform temperature on the surface of the slab. The findings of this work offer opportunities for improving the design of a broad range of engineering processes and products.

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Keywords

Solid slab with periodic array of isothermal strips; Heat conduction/convection; critical insulation thickness; Shape factor, total thermal resistance, overall heat transfer coefficient; Laplace equation.

Nomenclature

Bi	Biot number $Bi = hW/k$ (dimensionless)
G	Green's function of the temperature field (dimensionless)
H	One half the thickness of the slab (m)
h	Convection heat transfer coefficient ($\text{W}/\text{m}^2/\text{K}$)
k	Thermal conductivity ($\text{W}/\text{m}/\text{K}$)
L	Distance between two consecutive strips (m)
R_{tot}	Total thermal resistance (K/W)
S	Shape factor (m)
T	Temperature (K)
U	Overall heat transfer coefficient ($\text{W}/\text{m}^2/\text{K}$)
W	Width of a strip (m)
x, y	Coordinates of the physical plane (m)
z	Span of the strip (m)
<i>Diacritic</i>	
\wedge	The variable is normalized with the width of the strip W

1 Introduction

When an isothermal planar surface is covered with insulation, the total thermal resistance is always increased, and the effect is to reduce the energy dissipation [1, 2]. This result follows logically as one expects that the increase in the conduction path leads to an increase in the total resistance. However, in the case of cylindrical and spherical systems, adding this layer of insulation also increases the surface area available for heat transfer by convection. As there are two competing mechanisms that control the total rate of heat transfer, one expects that there is a critical radius such that the heat transfer rate is maximized. Indeed, it is a simple exercise to show that for cylindrical and spherical systems such a critical radius, where the total thermal resistance attains a minimum, exists and it is known as the critical insulation thickness [1]. This result is also applicable for surfaces with variable convection

coefficient, radiation loss and cylinders with transparent insulation [4, 5, 6]. The ideas can be also extended to non-circular domains: (i) square and rectangular domains etc. [2] where, using the shape factor, it is shown that a more general criterion for maximum heat dissipation is the critical perimeter of insulation, and (ii) domains with eccentric circular insulation where the maximum heat loss and the corresponding optimum insulation configuration must be found from the solution of the two-dimensional heat conduction equation with convective boundary condition imposed on the outside surface of the insulation [3]. In this work, similar to the latter approach [3], we show that the concept of critical insulation thickness can be also established for a planar, non-isothermal surface. The temperature variation is due to a finite, isothermal strip embedded below the surface that creates a non-trivial interaction between conduction through the slab and convection heat transfer from the planar surface.

Heat transfer in the slab-like configurations addressed here, is of interest in systems with distributed energy sources. In particular, high performance computing (e.g. laptops and desktop computers) requires multiple CPU units, each of which is a significant source of thermal energy. Thermal management in these systems is a topic of active research [7, 8, 9]. Moreover, both the problem formulated in this work and the results are relevant to a class of manufacturing processes related to thermal processing or operation of layered structures: (i) self-curing/bonding of laminate polymer matrix composites (PMC) where the heat is produced internally by conductive strips [10, 11, 12, 13], and (ii) internal (self) rapid thermal processing of semiconductor structures through embedded strips of nanoheaters [14, 15]. Besides heat transfer, many problems in modern science involve the solution of the Laplace equation, hence, the results of this study offer opportunities for improving the design of a broad range of engineering processes and products.

In the next Section (§2) we formulate an integral equation associated with the two-dimensional, heat conduction problem associated with a slab embedded with a periodic array of finite isothermal strips, and we obtain a Fredholm integral equation for the temperature gradient along a strip. The integral equation is solved both analytically (asymptotically) and numerically. The latter is discussed in Section 3. The results are also verified through

a finite-element numerical calculation. We summarize our findings in Section 4.

2 Problem Formulation and Asymptotic Results

Consider two-dimensional heat conduction due to a periodic array of isothermal (T_1) strips of infinite span, embedded at the center of a solid slab. The slab is subjected to convection heat transfer along its upper and lower surfaces as shown in Fig. 1. The strips of length W are placed a distance L apart. We non-dimensionalize lengths with the length of the strips (W), and the temperature field by subtracting T_∞ and normalizing by the temperature difference ($T_1 - T_\infty$). All nondimensional variables are denoted by $\hat{\cdot}$. In addition, because of symmetry, we consider only the upper half of the region, hence the formulation is also relevant for a periodic array of isothermal strips embedded in a non-conducting substrate [16].

At steady-state, the temperature distribution is governed by the Laplace equation $\nabla^2 \hat{T} = 0$. In view of the symmetry of the problem in the \hat{y} direction and periodicity in the \hat{x} direction, the boundary conditions are:

$$\begin{aligned} \text{On } \hat{y} = 0 \quad & \begin{cases} \hat{T}[\hat{x}] = 1 & \text{along } 0 \leq \hat{x} \leq 1 \\ \frac{\partial \hat{T}}{\partial \hat{y}}[\hat{x}] = 0 & \text{along } 1 < \hat{x} < \hat{L} \end{cases} \\ \text{On } \hat{y} = \hat{H} \quad & \frac{\partial \hat{T}}{\partial \hat{y}}[\hat{x}] + Bi \hat{T}[\hat{x}] = 0 \end{aligned} \quad (1)$$

$$\hat{T}[\hat{x} = 0, \hat{y}] = \hat{T}[\hat{x} = \hat{L}, \hat{y}],$$

where $Bi = hW/k$ defines the Biot number. The mathematical model along with the boundary conditions are shown in Fig. 2. The solution depends on the two geometric parameters, \hat{H} and \hat{L} , and the Biot number (Bi).

In the next section, using Green's theorem [17, 18, 19], we obtain a Fredholm integral equation of the first kind for the temperature gradient along a single strip. This equation is solved numerically, and asymptotic results for some limiting cases are also developed.

2.1 Green's Theorem formulation and asymptotic results

The objective of this section is to determine, as a function of \hat{L} , \hat{H} and Bi , the heat transfer rate (transport rate) from a single strip. Equivalently, we define the dimensionless Shape Factor (S) [20], the total thermal resistance (R_{tot}) and the overall heat transfer coefficient (U) [1] associated with a single strip as:

$$\begin{aligned} S &= -\mathbf{z} \int_0^1 \frac{\partial \hat{T}[\hat{x}, \hat{y} = 0]}{\partial \hat{y}} d\hat{x}, \\ R_{tot} &= \frac{1}{S k}, \\ U &= \frac{S k}{L \mathbf{z}}, \end{aligned} \quad (2)$$

respectively [1], where \mathbf{z} is the dimensional span of the strip. Note that we only consider the upper-half of the domain because of symmetry.

We can recast the problem into an integral equation along the boundary $\hat{y} = 0$, by applying Green's second identity [18, 21]. As outlined in Appendix A, an appropriate Green's function (G) is that associated with a periodic array of sources of period \hat{L} located along an insulated, lower surface [22] and a convective boundary condition along the upper surface:

$$\frac{\partial G[\hat{x}' - \hat{x}, \hat{y}' = 0]}{\partial \hat{y}'} = \sum_{m=-\infty}^{m=\infty} \delta[\hat{x}' - \hat{x} + m \hat{L}] = \frac{1}{\hat{L}} \sum_{k=-\infty}^{k=\infty} e^{2\pi i k (\hat{x}' - \hat{x}) / \hat{L}} \quad (3)$$

$$\frac{\partial G[\hat{x}' - \hat{x}, \hat{y} = \hat{H}]}{\partial \hat{y}'} + Bi G[\hat{x}', \hat{y}' = \hat{H}] = 0.$$

Employing the two-dimensional Green's second identity, we obtain an integral equation for the temperature gradient along the strip $0 \leq \hat{x} \leq 1$ (Appendix A):

$$1 = \int_0^1 \frac{\partial \hat{T}[\hat{x}', \hat{y}' = 0]}{\partial \hat{y}'} G[\hat{x}' - \hat{x}, \hat{y}' = 0] d\hat{x}', \quad (4)$$

where G is the Green's function associated with the boundary conditions (3). As shown in Appendix A, when evaluated at $\hat{y}' = 0$ we obtain (equation 25):

$$G[\hat{x}' - \hat{x}, \hat{y}' = 0] = -\frac{1 + \hat{H} Bi}{Bi \hat{L}} - \sum_{m=1}^{\infty} \frac{1}{m \pi} \frac{2 m \pi + Bi \hat{L} \tanh \left[\frac{2 m \pi \hat{H}}{\hat{L}} \right]}{Bi \hat{L} + 2 m \pi \tanh \left[\frac{2 m \pi \hat{H}}{\hat{L}} \right]} \cos \left[\frac{2 m \pi (\hat{x}' - \hat{x})}{\hat{L}} \right]. \quad (5)$$

In the next sections, we obtain asymptotic results of the integral equation for a number of cases.

2.1.1 The one-dimensional case ($\hat{L} = 1$)

When $\hat{L} = 1$, i.e. heat conduction between two infinite, flat plates, where one is kept at constant temperature and the other is subjected to convection, we obtain the trivial solution $\partial T[\hat{x}', \hat{y}' = 0]/\partial \hat{y}' = -Bi/(1 + Bi \hat{H})$ and the Shape Factor and the total thermal resistance are equal to

$$S[\hat{L} = 1] = z \frac{Bi}{1 + Bi \hat{H}}, \quad (6)$$

$$R_{tot}[\hat{L} = 1] = \frac{1 + Bi \hat{H}}{z Bi k}, \quad (7)$$

respectively. Consequently, R_{tot} increases monotonically with \hat{H} . With the exception of the edges, this result is approached for small values of the thickness (Fig. 3a).

2.1.2 Asymptotic result for small period ($\hat{L} \ll \hat{H}$)

When $\hat{L} \ll \hat{H}$, the terms $\tanh[2m\pi\hat{H}/\hat{L}] \approx 1$ for all m , and the infinite sum in equation (5) can be replaced by [23]

$$\sum_{m=1}^{\infty} \frac{1}{m} \cos \left[\frac{2m\pi}{\hat{L}} (\hat{x}' - \hat{x}) \right] = -\ln \left[2 \sin \left[\frac{\pi}{\hat{L}} |\hat{x}' - \hat{x}| \right] \right]. \quad (8)$$

If we further assume that $\hat{L} \gg 1$, then $\sin[\pi|\hat{x}' - \hat{x}|/\hat{L}] \approx \pi|\hat{x}' - \hat{x}|/\hat{L}$, and the integral equation (4) simplifies to:

$$1 = \int_0^1 \frac{\partial T[\hat{x}', \hat{y}' = 0]}{\partial \hat{y}'} \left(-\frac{1 + \hat{H} Bi}{Bi \hat{L}} + \frac{1}{\pi} \ln \left[\frac{2\pi}{\hat{L}} |\hat{x}' - \hat{x}| \right] \right) d\hat{x}'. \quad (9)$$

In what follows, we assume a solution of the form [24]

$$\frac{\partial \hat{T}[\hat{x}', \hat{y}' = 0]}{\partial \hat{y}'} = \frac{a_0}{\sqrt{\hat{x}'(1 - \hat{x}')}}, \quad (10)$$

which is also justified through, classic, boundary-layer analysis, i.e. the flux distribution has a square root singularity at the edge [18, 25]. This expression reduces the integral equation to an algebraic equation with solution

$$a_0 = -\frac{Bi \hat{L}}{\left(\pi + Bi \hat{H} \pi + Bi \hat{L} \ln[2 \hat{L}/\pi] \right)}. \quad (11)$$

This result is representative of the solution (Fig. 3b), and the square-root singularity is ubiquitous in these type of problems [1, 17, 18, 19, 22, 24, 25].

Hence, for $1 \ll \hat{L} \ll \hat{H}$ the total thermal resistance is predicted to be

$$R_{tot}[1 \ll \hat{L} \ll \hat{H}] = \frac{1/Bi + \hat{H} + \hat{L}/\pi \ln[2\hat{L}/\pi]}{zk\hat{L}}. \quad (12)$$

Again, R_{tot} increases monotonically with \hat{H} .

2.1.3 The case of a single strip ($\hat{L} \rightarrow \infty$, $Bi \gg 1$, $\hat{L} \gg 1$)

In the case of a single strip, the Fourier Transform of the Green's function (5) can be obtained by either taking the limit of equation (5) as $\hat{L} \rightarrow \infty$ [22], or by Fourier Transform techniques [24]:

$$\mathcal{F} \left[G[\hat{x}, \hat{y} = 0; \hat{L} \rightarrow \infty]; \hat{x} \Rightarrow \mathbf{w} \right] = -\frac{1}{\mathbf{w}} \frac{\mathbf{w} + Bi \tanh[\hat{H} \mathbf{w}]}{Bi + \mathbf{w} \tanh[\hat{H} \mathbf{w}]}. \quad (13)$$

The inverse of the above Fourier Transform is not listed, however, an approximate result can be obtained in the limit of large Bi as follows. The above equation is expanded for large Bi . The first term, i.e. $Bi \rightarrow \infty$, is simply the problem of heat conduction associated with an isothermal strip embedded in a slab with isothermal surfaces [24]. Hence, we take the inverse Fourier transform of the first two terms in the expansion to obtain

$$G[\hat{x}, \hat{y} = 0; \hat{L} \rightarrow \infty, Bi \gg 1] = -\sqrt{\frac{2}{\pi}} \ln \left[\coth \left[\frac{\pi \hat{x}}{4 \hat{H}} \right] \right] - \sqrt{\frac{\pi}{2}} \frac{\hat{x}}{Bi \hat{H}^2 \sinh[\pi \hat{x}/(2 \hat{H})]}, \quad (14)$$

where the first term is the result obtain in [24] associated with $Bi \rightarrow \infty$.

The above equation is further simplified if we assume that $\hat{H} \gg 1$, and the integral equation that governs the temperature gradient along the strip simplifies to:

$$1 = \frac{1}{\pi} \int_0^1 \frac{\partial T[\hat{x}', \hat{y}' = 0]}{\partial \hat{y}'} \left(\ln \left[\frac{\pi |\hat{x}' - \hat{x}|}{4 \hat{H}} \right] - \frac{1}{Bi \hat{H}} \right) d\hat{x}'. \quad (15)$$

Assuming a solution of the form [24]

$$\frac{\partial \hat{T}[\hat{x}', \hat{y}' = 0]}{\partial \hat{y}'} = \frac{a_0}{\sqrt{\hat{x}'(1 - \hat{x}')}}, \quad (16)$$

reduces the integral equation to an algebraic equation with solution

$$a_0 = -\frac{1}{1/(Bi \hat{H}) + \ln[16 \hat{H}/\pi]}. \quad (17)$$

The total thermal resistance is calculated to be

$$k z R_{tot}[\hat{L} \rightarrow \infty, Bi \gg 1, \hat{H} \gg 1] = \frac{1}{\pi} \left(\frac{1}{Bi \hat{H}} + \ln \left[\frac{16 \hat{H}}{\pi} \right] \right). \quad (18)$$

We note that this function has a minimum, $\partial R_{tot}/\partial \hat{H} = 0$ and $\partial^2 R_{tot}/\partial \hat{H}^2 > 0$, when $\hat{H} Bi = 1$ which, however, is not within the region of validity of the asymptotic expansion. Nevertheless, the asymptotic results suggest that there is a critical insulation thickness.

In the next section, we solve numerically the integral equation (4) associated with the heat conduction problem, compare the numerical results with the asymptotic results obtained in this section, and, finally, confirm the existence of a critical insulation thickness. Furthermore, a finite-element simulation has been performed using COMSOL [26] to verify the boundary element results.

3 Numerical Results

The integral equation (4) is solved numerically using the collocation boundary element method, i.e. the local basis functions are step functions, where the boundary ($0 \leq \hat{x} \leq 1$) is discretized into segments (boundary elements) and it is assumed that the unknown function is constant over each element [17, 18, 25]. In addition, we use Kummer's method [27, 28] to accelerate the convergence of the infinite summation in the Green's function (5) using the asymptotic result (8). This approach has been recently applied to address conduction problems in rectangular plates [29], where the solution may have very slowly convergent series, and is analogous to an Ewald summation method [30].

We first present some results for R_{tot} versus \hat{H} for various values of \hat{L} . As shown in Figs. 4 and 5, the numerical results approach the asymptotic results even for relatively moderate values of the respective, asymptotic parameters. In addition, the asymptotic expressions (7) and (12) can be regarded as upper and lower bounds of the total thermal resistance, respectively.

Most significant, the numerical results (Figs. 4 and 5) indicate that there exists a critical thickness such that the total thermal resistance associated with the slab attains a minimum, i.e. a maximum heat transfer rate can be achieved. This feature is considered further in Fig.

6, where we show the loci of critical thicknesses for different values of \hat{L} and Bi obtained through a one-parameter optimization procedure.

The boundary element results are further verified through a finite-element numerical simulation using COMSOL [26]. In the finite-element simulation, the domain has been truncated taking advantage of the symmetry in the \hat{x} -direction. In addition, the mesh has been refined close to the end point of the strip, as we expect that the temperature gradient would be singular at that particular point. Using a mesh of approximately 6500 elements we have managed to achieve an error of less than 1% as compared with the boundary element solution. In Fig. 7, we show a density/contour plot of the temperature field for a representative case. The finite element results compare well with the boundary element results.

A comparison of the two methods suggest that the boundary element is much more accurate and efficient, hence appropriate for the optimization problem associated with the critical thickness.

4 Conclusions

We have addressed the problem of heat conduction in a solid slab embedded with a periodic array of isothermal strips at the midsection. The upper and lower surfaces of the slab are subjected to convective heat transfer. The formulation is also relevant for a periodic array of isothermal objects placed on an insulated substrate.

The heat conduction problem, which is governed by the Laplace equation, is addressed using a Green's theorem approach to obtain a Fredholm integral equation of the first kind for the temperature gradient along a strip. The integral equation is solved numerically to obtain results for the heat transfer rate and the total thermal resistance, which compare well with asymptotic results. Most important, the numerical results reveal that there is a critical insulation thickness such that the total thermal resistance attains a minimum, i.e. a maximum heat transfer rate can be achieved. The concept of the critical thickness/radius is known to be applicable in cylindrical and spherical configurations, however the result has not previously been documented in planar configurations.

We have also performed finite-element calculations which compare well with the boundary-element results and yield further insight into the critical thickness. They reveal that, although the configuration under consideration is planar, the finite strips produce a non-trivial coupling between conduction and convection resulting in a 2D temperature distribution in the slab and a non-uniform temperature on the surface of the slab. This non-uniformity, along with a localized gradient near the tips of the strips, produce a balance between conduction and convection similar to cylindrical and spherical coordinates.

The findings of this work offer opportunities for improving the design of a broad range of engineering processes and products. Some specific examples are the design of systems with distributed energy sources such as high performance computers, and the curing and bonding of laminate matrix composites (PMC) where energy is produced internally through nano-heaters or internal resistances.

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A Appendix: Green's Function

In this section we develop an appropriate/convenient Green's function for the Laplace equation associated with the boundary conditions (1). The two-dimensional Green's second identity associated with the domain under consideration (Fig. 2), is ([31] pg 532, [21] pg 580):

$$\int_0^{\hat{L}} \left\{ G \left(-\frac{\partial \hat{T}}{\partial \hat{y}'} \right) - \hat{T} \left(-\frac{\partial G}{\partial \hat{y}'} \right) \right\}_{\hat{y}'=0} d\hat{x}' + \int_0^{\hat{L}} \left\{ G \left(\frac{\partial \hat{T}}{\partial \hat{y}'} \right) - \hat{T} \left(\frac{\partial G}{\partial \hat{y}'} \right) \right\}_{\hat{y}'=\hat{H}} d\hat{x}' = 0, \quad (19)$$

where the Green's function G , similar to \hat{T} , satisfies the Laplace equation. Employing the convective boundary condition (1), above equation simplifies to

$$\int_0^{\hat{L}} \left\{ G \left(-\frac{\partial \hat{T}}{\partial \hat{y}'} \right) + \hat{T} \left(\frac{\partial G}{\partial \hat{y}'} \right) \right\}_{\hat{y}'=0} d\hat{x}' - \int_0^{\hat{L}} \left\{ \hat{T} \left(Bi G + \frac{\partial G}{\partial \hat{y}'} \right) \right\}_{\hat{y}'=\hat{H}} d\hat{x}' = 0. \quad (20)$$

The above formulation suggests the following set of boundary conditions for G :

$$\begin{aligned} \frac{\partial G[\hat{x}', \hat{y}' = 0; \hat{x}]}{\partial \hat{y}'} &= \sum_{m=-\infty}^{m=\infty} \delta[\hat{x}' - \hat{x} + m \hat{L}] = \frac{1}{\hat{L}} \sum_{k=-\infty}^{k=\infty} e^{2\pi i k (\hat{x}' - \hat{x}) / \hat{L}} \\ &= \frac{1}{\hat{L}} \left(1 + 2 \sum_{k=1}^{k=\infty} \cos \left[\frac{2\pi k (\hat{x}' - \hat{x})}{\hat{L}} \right] \right) \end{aligned} \quad (21)$$

$$\frac{\partial G[\hat{x}', \hat{y}' = \hat{H}; \hat{x}]}{\partial \hat{y}'} + Bi G[\hat{x}', \hat{y}' = \hat{H}; \hat{x}] = 0,$$

which results to the integral equation:

$$T[\hat{x}, \hat{y} = 0] = \int_0^1 \frac{\partial \hat{T}}{\partial \hat{y}'}[\hat{x}', \hat{y}' = 0] G[\hat{x}' - \hat{x}, \hat{y}' = 0] d\hat{x}'. \quad (22)$$

When evaluated along the strip $0 \leq \hat{x} \leq 1$ we obtain an integral equation for the temperature gradient:

$$1 = \int_0^1 \frac{\partial \hat{T}}{\partial \hat{y}'}[\hat{x}', \hat{y}' = 0] G[\hat{x}' - \hat{x}, \hat{y}' = 0] d\hat{x}'. \quad (23)$$

To obtain the Green's function we assume a periodic solution of the form

$$G[\hat{x}', \hat{y}'; \hat{x}] = A_0[\hat{y}'] + \sum_{m=1}^{m=\infty} A_m[\hat{y}'] \cos \left[\frac{2m\pi(\hat{x}' - \hat{x})}{\hat{L}} \right]. \quad (24)$$

Substituting in the Laplace equation results to a system of ordinary differential equations which, along with the boundary conditions, leads to the Green's function. When evaluated at $\hat{y}' = 0$ we obtain:

$$G[\hat{x}' - \hat{x}, \hat{y}' = 0] = -\frac{1 + \hat{H} Bi}{Bi \hat{L}} - \sum_{m=1}^{\infty} \frac{1}{m \pi} \frac{2 m \pi + Bi \hat{L} \tanh\left[\frac{2 m \pi \hat{H}}{\hat{L}}\right]}{Bi \hat{L} + 2 m \pi \tanh\left[\frac{2 m \pi \hat{H}}{\hat{L}}\right]} \cos \left[\frac{2 m \pi (\hat{x}' - \hat{x})}{\hat{L}} \right]. \quad (25)$$

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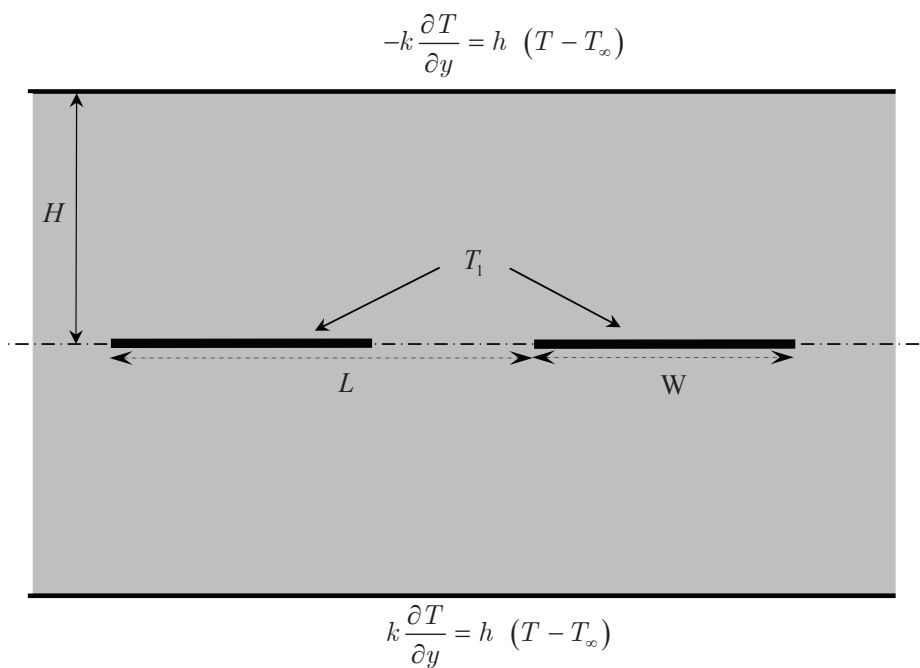


Figure 1: Schematic representation of the physical problem. The surfaces of the slab are subject to uniform convection, characterized by a heat transfer coefficient h , and the distance between them is $2H$. At the mid-distance there is a two-dimensional, periodic array of isothermal strips. The width of the strips is W and the distance between two successive strips is L . The strips are kept at temperature T_1 .

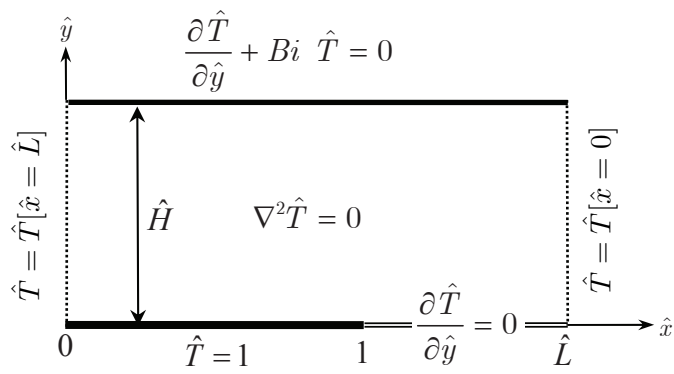


Figure 2: Schematic representation of the model problem along with boundary conditions. The Neumann boundary is dictated by the symmetry of the configuration.

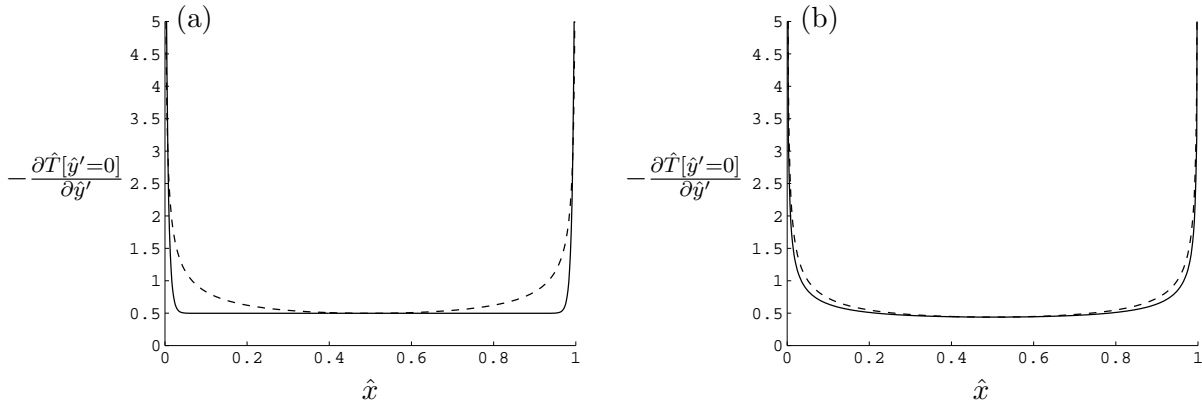


Figure 3: Computed values of the temperature gradient along the strip for $Bi = 0.5$ and $\hat{L} = 2$. The dashed curves corresponds to the asymptotic result (10) which implies a square-root singularity at the edges of the strip. (a) $\hat{H}=0.01$, (b) $\hat{H}=1$.

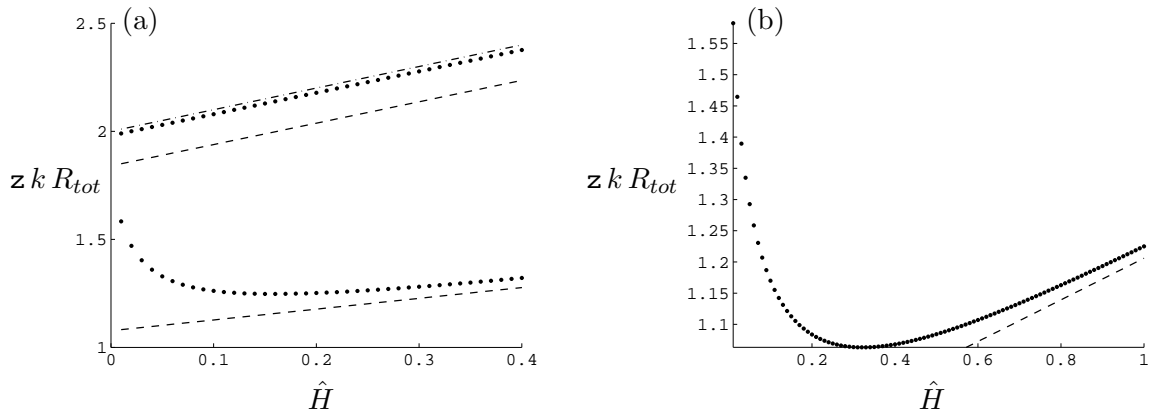


Figure 4: Computed values of the total thermal resistance as a function of \hat{H} for $Bi = 0.5$. The solid points correspond to numerical results and the dashed lines to expression (12) valid for $1 \ll \hat{L} \ll \hat{H}$. The dash-dot line in (a) corresponds to the asymptotic result (7) valid for $\hat{L} = 1$. (a) $\hat{L} = 1.01$ (upper dashed line and solid points) and $\hat{L} = 2$ (lower dashed line and solid points), (b) $\hat{L} = 3$.

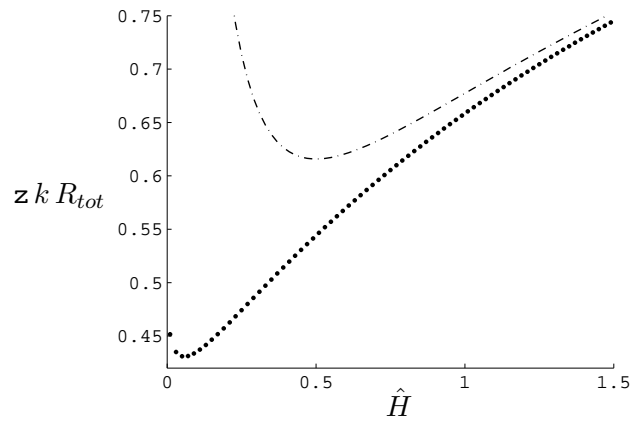


Figure 5: Computed values of the total thermal resistance as a function of \hat{H} for $Bi = 2$ and $\hat{L} = 20$. The solid points correspond to numerical results and the dashed curve to expression (18) valid for $\hat{L} \rightarrow \infty$, $Bi \gg 1$ and $\hat{H} \gg 1$.

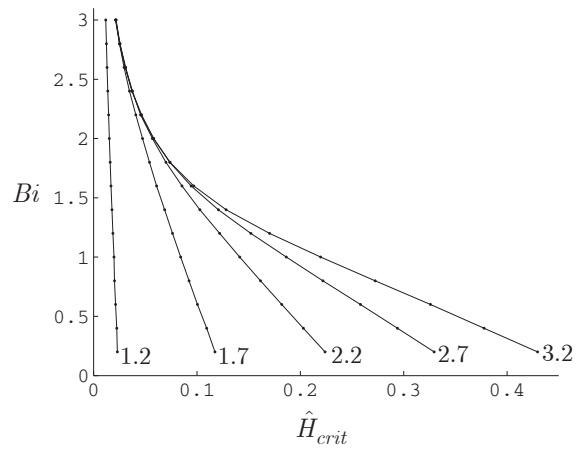


Figure 6: Curves of Biot number (Bi) versus critical thickness (\hat{H}_{crit}) for different values of the period \hat{L} . The value of \hat{L} is shown next to the curves.

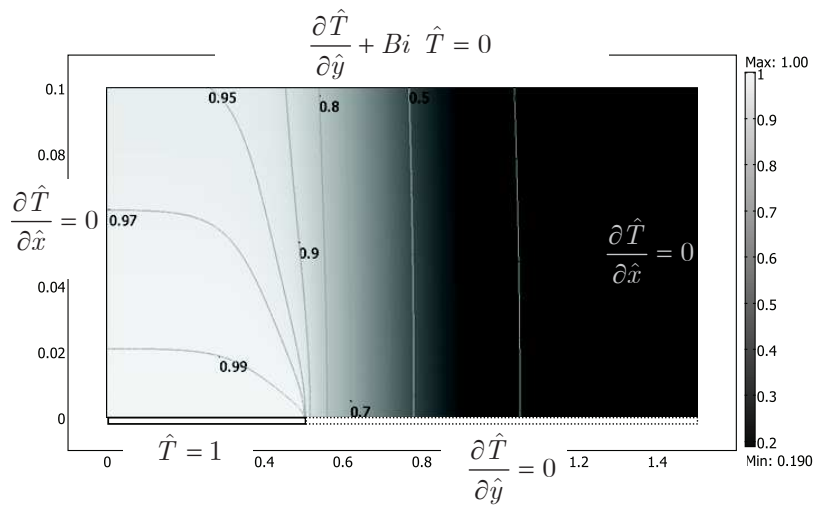


Figure 7: A density/contour plot of the temperature field obtained through finite element numerical simulations using COMSOL ($Bi = 0.5$). The domain and the boundary conditions are indicated. Note that, in addition to the symmetry of the domain in the \hat{y} -direction (Fig. 2), we have also taken advantage of the symmetry in the \hat{x} -direction.