Navigation of Multiple Kinematically Constrained Robots

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***Abstract*— In this paper, we propose a methodology for implementing multi-robot navigation function based controllers to mixed teams of holonomic and non-holonomic agents. A new non-smooth backstepping controller is introduced for translating kinematic controllers to equivalent dynamic ones, while maintaining bounded velocity specifications. The derived backstepping controller is applied to a dynamic model of the mobile robots, yielding a globally asymptotically stable dynamic controller. The effectiveness of the methodology is verified through non-trivial computer simulations.**

***Index Terms*— Multi-Robot Navigation, Nonholonomic systems, bounded input, non-smooth backstepping**

I. Introduction

Multi-Robot Navigation is a field of robotics that has recently gained increasing attention, due to the need to control more than one robot in the same workspace. The main motivation for our work initiated from the need to concurrently navigate several micro-robotic agents sharing the same constrained workspace.

The main focus of work on multi-robotic systems in the last few years has been on team formations [3], [4], [5], [6], [7], [8]. There have been several attempts to tackle multiagent navigation since the last two decades. [9], [10], [11], [12], [13], [14], [15]. Each one of them possess at least one of the following limitations:

- They are based on heuristic approaches
- They rely on simplifying assumptions i.e. point robots, convex obstacles, etc.
- They do not possess analytically guaranteed properties like stability, collision avoidance and global convergence
- They are not applicable for online trajectory generation
- They do not account for nonholonomic kinematics
- They do not consider bounded inputs
- They usually consider only the kinematic model of the system

In [9] the author defines separating planes at each moment and ensures that the robots stay in opposite half spaces but cannot guarantee that each robot will reach its goal since they may reach a deadlock state where one robot is blocking the other. In [10] a decoupled approach is presented, where first separate paths for the individual robots are computed and then possible conflicts of the generated paths are resolved in an off-line fashion. In [11] a categorization of multirobot navigation methodologies in centralized and decoupled planning methods is discussed. With centralized methods, one treats the separate robots as one composite robot transforming the multirobot problem to a single robot problem where standard motion planning methods can be used. The major drawback is that due to the large dimension of the configuration space the time complexity is very high. In decoupled planning methods the path of each robot is planned independently of the others and then a coordination scheme is applied to avoid mutual collisions. While this approach reduces the computational complexity, it is not complete in the sense that collision free paths cannot always be found even if they exist. In [12] maze searching techniques are applied to a multi-robot team to produce a decentralized multirobot motion planning scheme. However under the proposed model no provable motion planning strategy can be designed. An analysis and classification of existing multirobot coordination methodologies is presented in [13]. In first attempts to extend the navigation function concept to multi-robot setups, the authors of [14], [15] were able to establish collision free trajectories for a team of mobile robots. In [16] the authors consider an alternative problem in the domain of multirobot navigation, that is path coordination, where the robots paths are calculated off-line and a coordination scheme is executed in an off-line fashion. Although a large number of robots can be handled in this framework, it cannot handle inaccuracies in the executed trajectories, which are usually present in robotic systems due to the inability of the robots's hardware to follow exactly the pre-specified trajectory. In [17] a dynamic networks approach is adopted where a fast centralized planner is used to compute new coordinated trajectories on the fly. However this methodology does not have analytically guaranteed global convergence properties.

A need for a unifying framework for robotic navigation, where one can perform analysis and establish theoretical guarantees for the properties of the system is apparent. Such a framework was proposed by Koditschek and Rimon [18] in their seminal work. This framework had most of the qualities sought, but could only handle single point-sized robot navigation. The authors of this paper in their previous work [19] had contributed to successfully extending the navigation function framework to take into account the volume of each robot via appropriate transformations and also to handle robots with non-holonomic kinematic constraints. In this paper we

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make use of recent results on multi-robot navigation proposed by the authors of this paper in [20], as well as by the authors in [21]. More specifically we base our results on the new class of navigation functions, namely the multi-robot navigation functions that are proposed in those papers.

Of particular importance to multi-robot navigation is the case of systems possessing non-holonomic kinematic constraints. In [22] formation transitions of non-holonomic vehicle teams are studied using a graph theoretic approach. No general solutions have been proposed for closed loop navigation for multiple non-holonomic robots, due to the problem's complexity and the fact that non-holonomic systems do not satisfy Brocket's necessary smooth feedback stabilization conditions [23] hence no continuous static control law can stabilize a non-holonomic system to a point. Several motion planning strategies for non-holonomic systems are based on differential geometry [24], [25], [26], [27], [28], [29], [30]. Other strategies implement multi-rate [31] or time-varying controllers [32], [33]. Discontinuous control strategies are based on appropriately combining different controllers [34].

The main contributions of this paper can be summarized as follows:

1) A provably correct way to implement dipolar potential fields in Multi-Robot Navigation Functions for application in mixed holonomic and non-holonomic systems
2) Development of a Multi-Robot kinematic controller, that takes into account bounds in the maximum achievable velocities of the system
3) Development of a backstepping controller to translate the Multi-Robot kinematic controller to an equivalent dynamic controller, while maintaining the bounded velocity specifications

The rest of the paper is organized as follows: Section II introduces the considered system and presents the problem statement. Section III discusses the concept of multi-robot navigation functions, while in section IV the controller synthesis is presented. In section V simulation results of the proposed methodology are presented and the paper concludes with section VI.

II. System Description & Problem Statement

We consider m robots (figure 1) and assume that the n of them indexed $1 \dots n$ ($0 \leq n \leq m$) are holonomic, while the rest $z = m - n$ robots, indexed from $(n+1) \dots m$, are non-holonomic (see figure 2). Define the posture of each robot as

$$p_i = \left[q_i^T \theta_i\right]^T \in \mathbb{R}^2 \times (-\pi, \pi]$$

, with $i \in \{1 \dots m\}$, $q_i = \left[\begin{array}{cc} x_i & y_i \end{array}\right]^T$ is the position vector and θ_i the orientation of robot i. The state vector of the holonomic robots is defined as $\mathbf{p_h} = \left[p_1^T \dots p_n^T\right]^T$ and for the non-holonomic as $\mathbf{p_{nh}} = \left[p_{n+1}^T \dots p_m^T\right]^T$. The state of the whole system is $\mathbf{p} = \left[\mathbf{p_h}^T \mathbf{p_{nh}}^T\right]^T$ while the system orientation vector is $\theta = \left[\theta_1^T \dots \theta_m^T\right]^T$. For each holonomic agent the velocity vector is $\mathbf{\check{u}_{h_i}} = \left[\check{u}_{x_i} \check{u}_{y_i} \check{w}_i\right]^T \quad i = 1 \dots n$ while for each nonholonomic agent is $\mathbf{\check{u}_{nh_i}} = \left[\check{v}_i \check{w}_i\right]^T \ , \ i = (n+1) \dots m$. Thus the system velocity vector is $\mathbf{\check{u}} = \left[\mathbf{\check{u}}_h^T, \mathbf{\check{u}}_{nh}^T\right]^T$

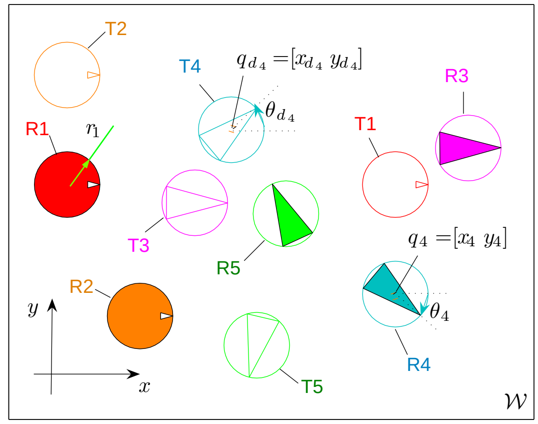

Fig. 1 *: Workspace populated with holonomic (filled disks: R1, R2) and non-holonomic (disks with filled triangle: R3, R4,R5) robotic agents. Target configurations represented with non-filled disks: T1,... T5.*

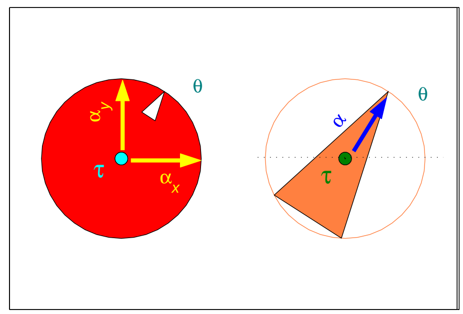

Fig. 2 *: Linear and angular velocities for a holonomic (a) and a non-holonomic (b) vehicle.*

with $\mathbf{\check{u}_h} = \left[\mathbf{\check{u}_{h_1}^T} \dots \mathbf{\check{u}_{h_n}^T}\right]^T$ and $\mathbf{\check{u}_{nh}} = \left[\mathbf{\check{u}_{nh_{n+1}}^T} \dots \mathbf{\check{u}_{nh_m}^T}\right]^T$. Assuming that

$$\left.\begin{array}{c} -\check{u}_{x_i}^{max} \leq \check{u}_{x_i} \leq \check{u}_{x_i}^{max} \\ -\check{u}_{y_i}^{max} \leq \check{u}_{y_i} \leq \check{u}_{y_i}^{max} \\ -\check{w}_i^{max} \leq \check{w}_i \leq \check{w}_i^{max} \end{array}\right\} \quad i = 1 \dots n$$

and

$$\left.\begin{array}{c} -\check{v}_i^{max} \leq \check{v}_i \leq \check{v}_i^{max} \\ -\check{w}_i^{max} \leq \check{w}_i \leq \check{w}_i^{max} \end{array}\right\} \quad i = (n+1) \dots m$$

we concatenate the row vectors $\left[\check{u}_{x_i}^{max} \; \check{u}_{y_i}^{max} \; \check{w}_i^{max}\right]$ for $i = 1 \dots n$ to obtain $\mathbf{\check{u}}_h^{max^T}$ and for $j = (n+1) \dots m$ the row vectors $\left[\check{v}_j^{max} \; \check{w}_j^{max}\right]$ to obtain $\mathbf{\check{u}}_{nh}^{max^T}$. Then

$$V = diag\left(\left[\mathbf{\check{u}}_h^{max^T} \quad \mathbf{\check{u}}_{nh}^{max^T}\right]\right)$$

is the diagonal matrix[1] with the maximum velocities achievable by the system. We can now consider $\mathbf{u} = \left[\mathbf{u}_h^T, \mathbf{u}_{nh}^T\right]^T = V^{-1}\mathbf{\check{u}}$ as the reduced[2] velocities vector . Obviously,

$$\begin{array}{rcl} \mathbf{u_h} & = & \left[\mathbf{u_{h_1}}^T \dots \mathbf{u_{h_n}}^T\right]^T \\ \mathbf{u_{h_i}} & = & \left[u_{x_i} u_{y_i} w_i\right]^T \\ \mathbf{u_{nh}} & = & \left[\mathbf{u_{nh_{n+1}}}^T \dots \mathbf{u_{nh_m}}^T\right]^T \\ \mathbf{u_{nh_i}} & = & \left[v_i w_i\right]^T \end{array} .$$

[1]operator $diag(x)$ returns a diagonal matrix whose main diagonal is the vector x.

[2]i.e. the velocities divided by their maximum values.

Consider the matrix

$$E = \begin{bmatrix} V_h & \underline{0} \\ \underline{0} & C \end{bmatrix}$$

with $V_h \in \mathbb{R}^{3n\times 3n}$ being the sub-matrix of V containing the maximum velocities achievable by the holonomic subsystem and

$$C = \begin{bmatrix} C_{n+1} & & 0 \\ & \ddots & \\ 0 & & C_m \end{bmatrix}$$

with

$$C_i = \begin{bmatrix} \check{v}_i^{max} cos(\theta_i) & 0 \\ \check{v}_i^{max} sin(\theta_i) & 0 \\ 0 & \check{w}_i^{max} \end{bmatrix}$$

since we are modeling the non-holonomic systems as non-holonomic unicycles.

After those considerations the mixed-robotic team kinematics can be described by:

$$\dot{\mathbf{p}} = E \cdot \mathbf{u} \tag{1}$$

while, based on the planar motion assumption, the dynamics can be derived using the Lagrange–d'Alembert equations [35] and is of the form:

$$\mathbf{M}\dot{\check{\mathbf{u}}} + \mathbf{R}\left(\mathbf{p}, \check{\mathbf{u}}\right) = f \tag{2}$$

where $\mathbf{M}$ is the inertia matrix, matrix $\mathbf{R}$ contains the Coriolis and centrifugal terms and f is the vector of input forces and torques. Equation (2) along with (1) determine the dynamics of the system. The considered upper bounds to the robots' achievable velocities are reflected in the following restrictions over the velocity inputs to system (1):

$$|u_{x_i}| \leq 1, |u_{y_i}| \leq 1, |w_i| \leq 1, \; i \in \{1 \ldots n\} \tag{3a}$$

$$|w_i| \leq 1, |v_i| \leq 1, \; i \in \{n+1 \ldots m\} \tag{3b}$$

The problem we are considering, can be stated as follows: *Given the mixed holonomic and non-holonomic system ((2) & (1)) and the velocity constraints (3), derive a feedback control law that steers the system from any initial configuration to the goal configuration avoiding collisions. The environment is assumed perfectly known and stationary.*

III. Multi-Robot Navigation Functions (MRNFs)

A. Preliminaries

Navigation Functions (NFs) are real valued maps, realized through cost functions, the negated gradient fields of which are attractive towards the goal configuration and repulsive with respect to obstacles. Considering a trivial system described kinematically as $\dot{q} = u$ the basic idea behind navigation functions is to use a control law of the form $u = -\nabla\varphi(q)$ where $\varphi(q)$ is a navigation function, to drive the system to its destination (figure 3). It has been shown (Koditschek and Rimon [18]) that strict global navigation (i.e. with a globally attracting equilibrium state) is not possible and a smooth vector field on any sphere world, which has a unique attractor, must have at least as many saddles as obstacles. The fundamental concern during design is assuring that the critical points (excluding destination point which is a global minimum) comprise a set of measure zero of saddle points. The assumption about spherical robots and obstacles does not constrain the generality of this work since it has been proven [18] that navigation properties are invariant under diffeomorphisms. Methods for constructing analytic diffeomorphisms are discussed in ([36],[37]) for point robots and in [38] for rigid body robots.

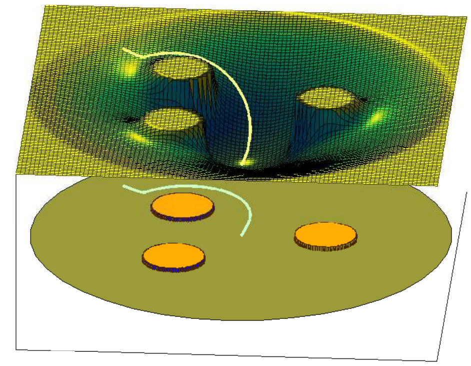

Fig. 3 *: Navigation Function with three obstacles and the resulting gradient following path.*

Let us assume the following situation: We have m mobile robots, and their workspace $\mathcal{W} \subset \mathbb{R}^r$ where r is the workspace dimension. Each robot occupies a sphere:

$$R_i \triangleq \{\mathbf{q} \in \mathbb{R}^r : \|\mathbf{q} - \mathbf{q}_i\| \leq r_i\}, \; i = 1 \ldots m,$$

where $\mathbf{q}_i \in \mathbb{R}^r$ is the robot position and r_i is the sphere radius. The configuration of each robot is represented by position $\mathbf{q}_i$ and the configuration space C is spanned[3] by $\mathbf{q} = \left[\mathbf{q}_1^T \ldots \mathbf{q}_m^T\right]^T$. The destination configurations are denoted with the index d, i.e. $\mathbf{q}_d = \left[\mathbf{q}_{d_1}^T \ldots \mathbf{q}_{d_m}^T\right]^T$. Figure 1 depicts a team of robots, where the holonomic are represented as filled disks while the nonholonomic are represented as disks with filled triangles in a spherical workspace. A multi-robot navigation function can be defined in an analogous manner to the navigation function definition [18] as follows:

Definition 1: [1] Let $\mathcal{F} \subset \mathbb{R}^{rm}$ be a compact connected analytic manifold with boundary. A map $\varphi : \mathcal{F} \to [0, 1]$ is a multirobot navigation function if it is:

1) Smooth on $\mathcal{F}$,
2) Polar on $\mathcal{F}$, with minimum at $\mathbf{q}_d \in \overset{o}{\mathcal{F}}$,
3) Morse on $\mathcal{F}$,
4) $\lim_{q \to \partial\mathcal{F}} \varphi(q) = 1 > \varphi(q_{int}), \; \forall q_{int} \in \overset{o}{\mathcal{F}}$.

Regarding continuity, multirobot navigation functions must be at least C^2. The property 1 of Definition 1 follows the intuition provided by the authors of [18], that is preferable to use analytic functions to avoid branching and looping in the control algorithm and to provide a provably correct control algorithm for every environment that could be diffeomorphically transformed to a sphere world.

A function φ is called polar if it has a unique minimum on $\mathcal{F}$. By using smooth vector fields one cannot do better than

[3] The MRNF design takes place in a sphere world, where the robots are orientation free, hence orientation will be considered later in the controller synthesis.

have almost global navigation [18]. By using a polar function on a compact connected manifold with boundary, all initial conditions will either be brought to a saddle point or to the unique minimum: $\mathbf{q}_d$.

A scalar valued function φ is called a Morse function if all its critical points (zero gradient vector field) are non-degenerate, that is its hessian at the critical points is full rank. The requirement in Definition 1 that a navigation function must be a Morse function, establishes that the initial conditions that bring the system to saddle points are sets of measure zero [39]. In view of this property, all initial conditions away from sets of measure zero are brought to q_d.

The last property of definition 1 guarantees that the resulting vector field is transverse to the boundary of $\mathcal{F}$. This establishes that the system will be safely brought to q_d, avoiding collisions.

We are currently aware of two examples of multi-robot navigation function constructions that abide by Definition 1: The one proposed by the authors in [1], [20], [40] and the one proposed in [14], [15], [21]. We note here that for the proposed techniques in this paper, one can consider any of the above mentioned MRNF constructions. In the following we will present some background material regarding multi-robot navigation functions.

B. NFs vs MRNFs

The concept behind potential functions is that the system must be attracted toward the "good" sets and repelled away from "bad" sets. Multi-Robot Navigation functions are a special category of potential functions that have the properties defined in Definition 1. The navigation function φ proposed by Koditschek and Rimon [18] for single, point robot navigation, is a composition of three functions:

$$\varphi \triangleq \sigma_d \circ \sigma \circ \hat{\varphi} = \frac{\gamma_d}{\left(\gamma_d^k + \beta\right)^{1/k}} \tag{4}$$

where $\sigma_d(x) \triangleq x^{1/k}$, $\sigma(x) \triangleq \frac{x}{1+x}$, $\hat{\varphi} = \frac{\gamma}{\beta}$. Function $\sigma_d(\cdot)$ is used to render the destination point a non-degenerate critical point, function $\sigma(\cdot)$ is used to constrain the values of the navigation function in the range of $[0,1]$ and in function $\hat{\varphi}$, $\gamma = \gamma_d^k = \|q - q_d\|^{2k}$ is a metric of the distance from the target - hence the good set is defined as $\gamma^{-1}(0)$ while the bad sets are defined as $\beta^{-1}(0)$. Now the essential difference between single point robot and multiple non-point robot navigation lies in the way of choosing function β. For the single point robot case, this function was chosen as the product of the functions β_j that encoded class K_∞ functions of the distance of the robot from the obstacles and the workspace boundary. In initial attempts to tackle the non-point multi-robot navigation problem in the context of navigation functions, the authors of [14], [15] chose function β as the product of the functions $\beta_{i,j} = \frac{1}{2}\left(\|q_i - q_j\|^2 - (\rho_i + \rho_j)^2\right)$ encoding metrics of the distances of robot pairs yielding a polynomial computational complexity with respect to the number of robots. Initial results provided for theoretical guaranteed collision avoidance while in more recent results [21] it is proven that the proposed construction is actually a (multi-robot) navigation function.

In the approach presented in [1], [20], [40] a more elaborate construction of β is considered that is encoding a measure of the distance of the robot positions from all the possible collision arrangements, resulting in an exponential computational complexity with respect to the number of considered robots.

IV. Controller Synthesis

A. Dipolar MRNFs

A navigation field with dipolar like structure is particularly suitable for nonholonomic navigation [41]. Figure 4 shows a 2D dipolar Navigation Function. The idea here is to create flow lines that resemble that of a dipole, so that the system trajectories approach the destination with orientation perpendicular to the dipole axis. In order to apply the dipolar navigation methodology to the problem we are considering, the β function of the MRNF (4) must be modified to get an MRNF as follows:

$$\varphi = \frac{\gamma_d}{\left(\gamma_d^k + H_{nh} \cdot \beta\right)^{1/k}} \tag{5}$$

where H_{nh} has the form of a pseudo - obstacle:

$$H_{nh} = \varepsilon_{nh} + \prod_{i=n+1}^{m} \eta_{nh_i}.$$

The term γ_d is as defined in the previous section. The term

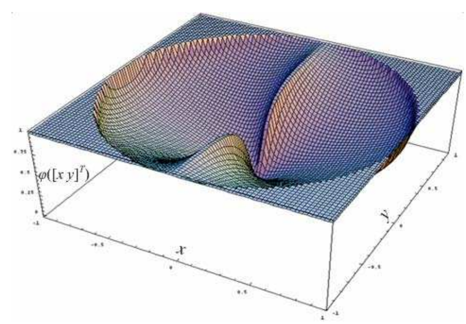

Fig. 4 *: 2D Dipolar Navigation Function.*

β depends on which construction we use, i.e. it could either be the one proposed in [1] or the one proposed in [14]. The navigation properties are not affected by this modification, as long as the workspace is bounded, η_{nh_i} can be bounded in the workspace and $\varepsilon_{nh} > \varepsilon_{\min} > 0$ [42], [2]. A possible choice of η_{nh_i} is:

$$\eta_{nh_i} = \left((\mathbf{q} - \mathbf{q}_d)^T \cdot \mathbf{n}_{d_i}\right)^2 \tag{6}$$

where $\mathbf{n}_{d_i} = \left[\underline{0}_{1\times 2(i-1)}\ \cos(\theta_{d_i})\ \sin(\theta_{d_i})\ \underline{0}_{1\times 2(m-i)}\right]^T$. Alternative choices of η_{nh_i} can be implemented, e.g. the one proposed in [43]. η_{nh_i} defined in (6) denotes the squared distance of the robot from the pseudo-obstacle, which is oriented perpendicular to the robot's destination configuration. This "pushes" the robot away from positions that are oriented across the pseudo-obstacle creating in this way a more reasonable trajectory for a non-holonomic system to follow.

B. Design

Using the MRNF $\varphi(\mathbf{q})$ we can construct the following Lyapunov function candidate:

$$V(\mathbf{p}) \triangleq \varphi(\mathbf{q}) + \frac{1}{2}\lambda_1 \theta^T \theta$$

with $\lambda_1 > 0$. We denote with $V_x,\ V_y,\ V_\theta$ the derivatives of V along the $x,\ y$ and θ directions respectively.

Define the set of the z non-holonomic indices:

$$\mathcal{M} \triangleq \{n+1,\ldots,m\}$$

and

$$\Omega \triangleq P(\mathcal{M})$$

where P denotes the power set operator. Assuming that Ω is an ordered set, let N_j denote the j 'th element of Ω where $j \in \{1,\ldots,2^z\}$. Then $N_j \subseteq \mathcal{M}$ with $N_1 = \{\emptyset\}$ and $N_{2^z} = \mathcal{M}$. We can now define:

$$\Delta_j \triangleq \delta_{\theta_{nh}}(j) - \delta_{V_q} - \delta_h \tag{7}$$

where $\delta_{\theta_{nh}}, \delta_{V_q}, \delta_{V_\theta}$ are defined as follows:

$$\delta_{\theta_{nh}}(j) \triangleq \sum_{i\in\{M\backslash N_j\}} \left[\tau \frac{(\theta_{nh_i}-\theta_i)\cdot \breve{w}_i^{max}\cdot V_{\theta_i}}{a_1 + |\theta_{nh_i}-\theta_i|}\right] - \\ - \sum_{i\in\{N_j\}} \left[\tau \frac{\breve{w}_i^{max}\cdot V_{\theta_i}^2}{a_1+|V_{\theta_i}|}\right]$$

$$\delta_{V_q} \triangleq \tau \sum_{i=n+1}^{m} \left[|V_{x_i}\cdot cos(\theta_i) + V_{y_i}\cdot sin(\theta_i)| \cdot \frac{\breve{v}_i^{max}\cdot Z_i}{a_2+Z_i}\right]$$

$$\delta_h \triangleq \tau \sum_{i=1}^{n} \frac{\breve{u}_{x_i}^{max}\cdot V_{x_i}^2}{a_1+|V_{x_i}|} + \frac{\breve{u}_{y_i}^{max}\cdot V_{y_i}^2}{a_1+|V_{y_i}|} + \frac{\breve{w}_i^{max}\cdot V_{\theta_i}^2}{a_1+|V_{\theta_i}|}$$

$$Z_i \triangleq \left(a_3\cdot\left(V_{x_i}^2+V_{y_i}^2\right) + a_4\left((x_i-x_{d_i})^2+(y_i-y_{d_i})^2\right)\right)^{1/2}$$

is a mixed metric of the distance to the target and the gradient level,

$$\theta_{nh_i} \triangleq atan2\left(V_{y_i}\cdot side_i, V_{x_i}\cdot side_i\right)$$

is the "non-holonomic" angle we want the system to track where

$$side_i \triangleq sgn\left((\mathbf{q}-\mathbf{q}_d)\cdot \mathbf{n}_{d_i}\right)$$

denoting on which side of the pseudo-obstacle the robot is with

$$sgn(x) \triangleq \begin{cases} -1 & x<0 \\ 1 & x\geq 0 \end{cases}$$

and $a_1,\ a_2,\ a_3,\ a_4$ are positive constants. Define

$$H \triangleq \{j : \Delta_j < 0\}$$

and

$$\rho \triangleq \min\left\{H \bigcup \{2^z\}\right\}.$$

Also define

$$s(x) \triangleq \frac{x}{a_1+|x|}.$$

We can now state the following kinematic control law:

Proposition 1: The system (1) under the control law:

$$\begin{array}{rcl} u_{x_\ell} &=& -\tau\cdot s(V_{x_\ell}) \\ u_{y_\ell} &=& -\tau\cdot s(V_{y_\ell}) \ ,\ \ell\in\{1,\ldots n\} \\ \omega_\ell &=& -\tau\cdot s(V_{\theta_\ell}) \end{array}$$

$$\begin{array}{rcll} \omega_l &=& -\tau\cdot s(\theta_l-\theta_{nh_l}), & l\in\{\mathcal{M}\backslash N_\rho\} \\ \omega_j &=& -\tau\cdot s\left(V_{\theta_j}\right), & j\in\{N_\rho\} \end{array}$$

$$v_i = -\tau\cdot\frac{Z_i}{a_2+Z_i}\cdot \mathrm{sgn}\left(V_{x_i}\cdot\cos(\theta_i)+V_{y_i}\cdot\sin(\theta_i)\right), \quad i\in\mathcal{M}$$

where $0<\tau\leq 1$, is globally asymptotically stable a.e.[4]

Proof: See Appendix II-A. ■

Corollary 1: The control law defined in Proposition 1 respects the input constraints defined in (3).

Proof: Since the range of function

$$-1\leq s(x)\leq 1, \forall x\in\mathbb{R}$$

and

$$|u_i| = \tau\frac{Z_i}{a_2+Z_i}\leq\tau\leq 1,$$

the constraints (3) are not violated. ■

C. Backstepping Kinematics into Dynamics

The controller proposed in Proposition 1 is a kinematic feedback control law that stabilizes (1). In order to stabilize the dynamic system (1 & 2) we need to translate the kinematic controller to an appropriate dynamic controller.

Choosing the vector of input forces as

$$f = \mathbf{R}(\mathbf{p}, V\mathbf{u}) + \mathbf{M}(\mathbf{p})V\nu \tag{8}$$

linearizes (2) via feedback. The feedback linearized form of system (2 & 1) is:

$$\dot{\mathbf{p}} = E\cdot\mathbf{u} \tag{9a}$$
$$\dot{\mathbf{u}} = \nu \tag{9b}$$

Let $|x|$ denote a vector whose elements are the absolute values of the respective elements of x. $\mathbf{1}_m$ is the $m\times 1$ vector whose elements are ones and I_m is the $m\times m$ identity matrix. V^o, ∂V, are the generalized directional derivative and generalized gradient of V respectively and are defined in Appendix I. The generalized time derivative of V, $\dot{\tilde{V}}$ is defined in Theorem II-1 in the appendix II-A. Solutions of differential equations with discontinuous right hand sides are defined in terms of a differential inclusion $\mathbf{F}(\cdot)$ according to [44].

The proposed backstepping controller to translate the kinematic controller to an equivalent dynamic controller, is based on the non-smooth backstepping controller design proposed in [45] which is extended in this paper so that the constraints (3) are satisfied:

Theorem 1: Consider the system:

$$\dot{\eta} = g(\eta)\xi \tag{10a}$$
$$\dot{\xi} = \nu \tag{10b}$$

[4]a.e.: almost everywhere, i.e. everywhere except a set of initial conditions of measure zero that lead the holonomic subsystem to saddle points

where $\eta \in \mathbb{R}^n$, $\xi \in \mathbb{R}^m$, $\mathbb{R}^{n\times m} \ni g(\eta) \neq \underline{0}$. Assume that the subsystem (10a) can be stabilized by a control law $\xi = \phi(\eta)$ with $\phi(Q) = 0$ where $Q = \{\eta_d\} \bigcup \{\mu N\}$ where η_d the destination configuration and μN a set of measure zero of saddle points and the set of initial conditions that lead to μN are of measure zero, and $\|\phi(\eta)\|_\infty \leq \tau < 1$, and that there is a (Multi-Robot) Navigation Function $V(\eta)$. Then the following control law:

$$\nu = \sigma - \mu \tag{11}$$

where

$$\sigma = \zeta + \left(K_z + \frac{V^o(\eta; g(\eta)[\xi - \phi(\eta)])}{\|\xi - \phi(\eta)\|^2} I_m \right) \cdot [\phi(\eta) - \xi] \tag{12}$$

and

$$\mu = diag\left(a|\xi| + b\mathbf{1}_m\right) \cdot diag\left(\mathbf{1}_m - |\xi|\right)^{-1} \cdot (\xi - \phi(\eta)) \tag{13}$$

where ζ is the minimum norm element of $\dot{\tilde{\phi}}$, K_z a positive definite constant matrix, a, b positive constants such that

$$0 < b < 1 - \tau$$

and

$$a = \frac{1 - \tau - b}{\tau},$$

globally asymptotically stabilizes (10) a.e., while satisfying $\|\xi\|_\infty < 1$.

Proof: See Appendix II-B. ∎

We can now state the following:

Proposition 2: System (1 & 2) with the vector of input forces (8), where ν is the control law defined in (11) with ϕ the kinematic control law as defined in Proposition 1 and the constants a, b as defined in Theorem 1 is:

- globally asymptotically stable a.e.
- satisfies the velocity constraints (3).

Proof: Using (8) we get the system in the form (9). Setting

$$\eta \triangleq \mathbf{p}, \quad g(\eta) \triangleq \begin{bmatrix} V_h & \underline{0} \\ \underline{0} & C \end{bmatrix}$$

and

$$\xi \triangleq \mathbf{u},$$

system (9) is reduced to system (10). By Corollary 1 we have that

$$\|\phi\|_\infty \leq \tau < 1.$$

Applying Theorem (1) and noting that $\|\mathbf{u}\|_\infty < 1$ satisfies the constraints (3) completes the proof. ∎

V. Simulations

To verify the effectiveness of our algorithms, we have set-up a simulation with 5 robots, with the multirobot navigation function construction proposed in [20]. The robots are represented as circles with an inscribed triangle indicating their current orientation. Holonomic robots were represented as filled disks and non-holonomic robots as disks with an inscribed filled triangle (figure 1).

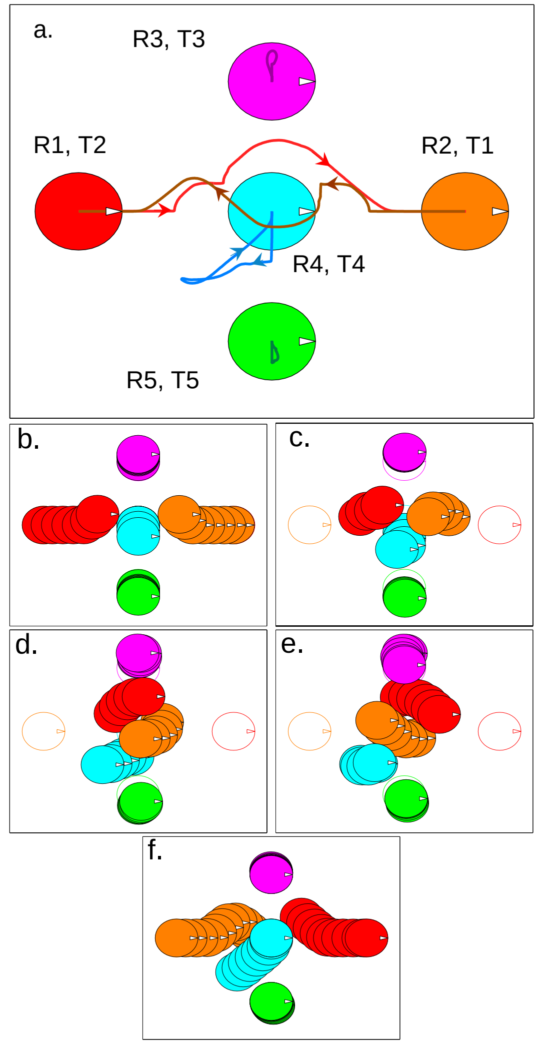

Fig. 5 *: First simulation with* 5 *holonomic robots.*

In the first simulation we used only holonomic robots to demonstrate the effectiveness of the proposed controller with multirobot navigation functions. The initial robot configurations, indicated with Ri and their target configurations Ti, with $i \in \{1 \ldots 5\}$ are shown in figure 5-a. Robots 1 and 2 were initially placed at each others target, whereas robots $3 \ldots 5$ were initially placed at their destination configurations. The same figure (5-a) depicts the path that each robot followed during the simulation. The rest of the snapshots of figure 5 depict the evolution of the system. Observe how robots $3 \ldots 5$ move away from their targets to allow for robots 1 and 2 to maneuver their way to their targets. Eventually all robots converge to their targets. In figure 6 the velocities for each robot are shown. Since initial and final angles are identical for the holonomic simulation, the angular velocities are zero. As can be seen from figure 6, the velocity for each actuation direction lies in the predefined velocity bounds indicated by the dotted lines at $\pm 100\%$ of maximum allowable velocity levels. Note here that since the allowable velocities are an invariant set for the backstepping controller, as indicated in Lemma 1 in Appendix II-B and the bounding is performed through a non-linear scaling, the velocities do not saturate but

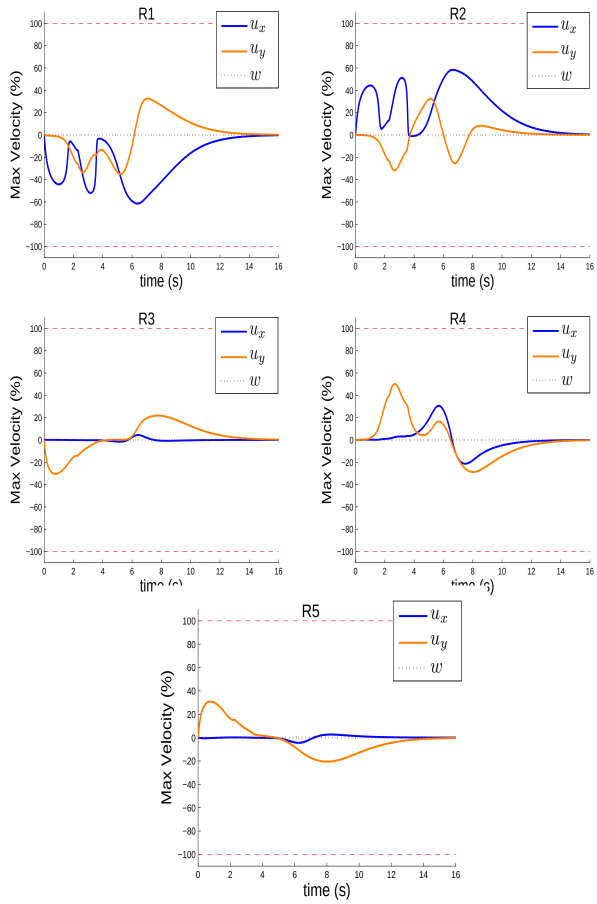

Fig. 6 : Robot velocities for the first simulation for each robot.

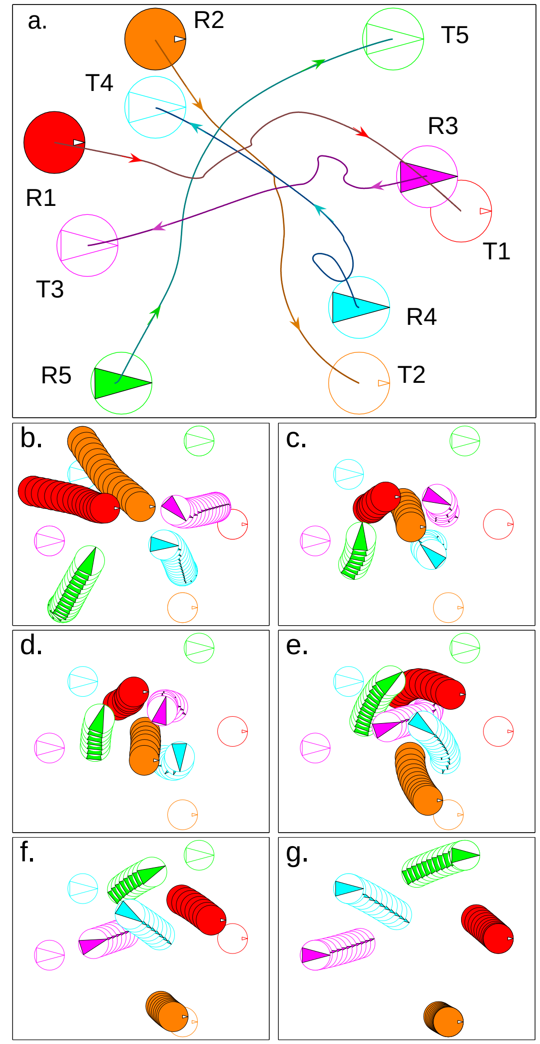

Fig. 7 : Second simulation with 2 holonomic and 3 non-holonomic robots.

evolve inside the predefined velocity bounds.

In the second simulation we used 2 holonomic robots ($R1$, $R2$) and 3 non-holonomic robots $R3 \ldots R5$ to show the effectiveness of dipolar multirobot navigation functions in scenarios with mixed holonomic - non-holonomic robot teams (a video of this simulation will be available at http://ieeexplore.ieee.org). The initial and final robot configurations indicated as Ri, Ti resp., with $i \in \{1 \ldots 5\}$ are shown in figure 7-a. The same figure (7-a) depicts the path followed by each robot during the simulation. Figures 7-b to 7-g depict snapshots of the robot trajectories and maneuvers during the system evolution. In this mixed scenario the multirobot navigation functions augmented with an appropriate dipolar structure succeed in navigating the mixed robotic team to its destination. Figure 8 depicts the control effort for each robot. As can be seen from figure 8, the non-smooth backstepping controller acts as a low pass filter suppressing the discontinuities and the chattering behavior that exists in the non-holonomic discontinuous kinematic controller, while keeping the system velocities within the predefined velocity bounds. Note that due to the closed form feedback solution we have a very fast feedback controller. Typically it takes about 10 milliseconds on a 700 MHz Pentium III processor to calculate the control signal without any code optimizations.

VI. Conclusion

A methodology for implementing multi-robot navigation functions to teams of mixed holonomic and non-holonomic robots with full dynamics and with constraints on their achievable velocities was presented in this paper. The multi-robot navigation functions are augmented with a dipolar structure that is suitable for non-holonomic navigation. The proposed controllers provide upper bounded velocities to the system, while maintaining the MRNF's global convergence and collision avoidance properties. The methodology due to its closed loop nature provides a robust navigation scheme with guaranteed collision avoidance and its global convergence properties guarantee that a solution will be found if one exists. The closed form control law provides fast feedback making the methodology suitable for real time applications.

Future research directions include the extension of this technique to decentralized systems, along the lines of [46], but with limited sensing and communication capabilities, and the study of multi-robot navigation in obstacle cluttered environments and more specifically to derive efficient transformations to map obstacle cluttered environments to equivalent spherical ones,

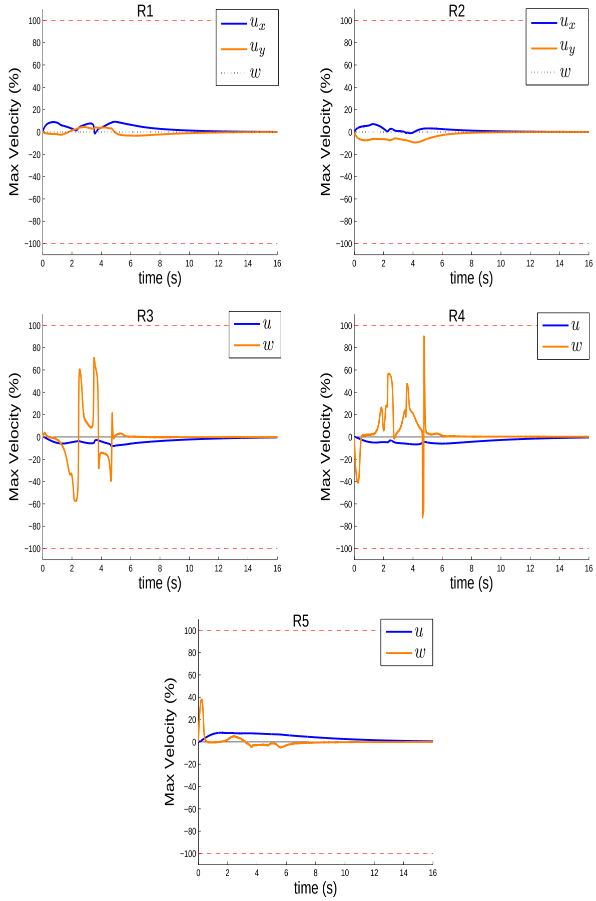

Fig. 8 : Robot velocities for the second simulation for each robot.

that are appropriate for multi-robot navigation.

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Appendix I
Definitions

This section contains several definitions used in this paper.

Definition I-1: [47] Let f be Lipschitz near a given point x and let v be any other vector in $\mathbf{X}$. The generalized directional derivative of f at x in the direction v, denoted $f^o(x;v)$, is defined as follows:

$$f^o(x;v) \triangleq \limsup_{\substack{y\to x \\ t\to 0}} \frac{f(y+tv)-f(y)}{t}$$

where y is a vector in $\mathbf{X}$ and t a positive scalar.

Definition I-2: [47] Let f be Lipschitz near x. The generalized gradient of f at x, denoted $\partial f(x)$ is the subset of [the dual space of $\mathbf{X}$] $\mathbf{X}^*$, given by:

$$\partial f(x) \triangleq \{\zeta \in \mathbf{X}^* \; f^o(x;v) \geq \langle \zeta, v\rangle, \forall v \in \mathbf{X}\}$$

where $f^o(x;v)$ is the generalized directional derivative defined in Definition I-1.

Appendix II

A. Proof of Proposition 1.

Since the control scheme we are considering is discontinuous, the right hand side of (1) is discontinuous hence we need to consider the Filippov sets created over the switching regions. We will need the following results from non-smooth analysis:

Definition II-3: ([44]) A vector function x is called a solution of

$$\dot{x} = f(x)$$

if x is absolutely continuous and

$$\dot{x} \in \mathcal{K}[f](x)$$

where

$$\mathcal{K}[f](x) \triangleq \overline{co}\{\lim f(x_i) \,|x_i \to x, \; x_i \notin N\}$$

where N is a set of measure zero.

Theorem II-1: [48] Let $x(\cdot)$ be a Filippov solution to

$$\dot{x} = f(x)$$

and $V : \mathbb{R}^m \to \mathbb{R}$ be a Lipschitz and regular function. Then $V(x)$ is absolutely continuous, $\frac{d}{dt}V(x)$ exists almost everywhere and

$$\frac{d}{dt}V(x) \in^{a.e.} \dot{\tilde{V}}$$

where

$$\dot{\tilde{V}} \triangleq \bigcap_{\xi \in \partial V(x)} \xi^T \cdot \mathcal{K}[f](x)$$

is the generalized time derivative of V and ∂V is the Clarke's generalized gradient [47].

The following theorem is an extension to LaSalle's invariance principle for non-smooth systems:

Theorem II-2: [48] Let Ω be a compact set such that every Filippov solution to the autonomous system $\dot{x} = f(x), \; x(0) = x(t_0)$ starting in Ω is unique and remains in in Ω for all $t > t_0$. Let $V : \Omega \to \mathbb{R}$ be a time independent regular function such that $v \leq 0$ for all $v \in \dot{\tilde{V}}$. (If $\dot{\tilde{V}}$ is the empty set then this is trivially satisfied). Define

$$S = \left\{x \in \Omega | 0 \in \dot{\tilde{V}}\right\}$$

. Then, every trajectory in Ω converges to the largest invariant set, M in the closure of S.

Function V is a regular function, since it's smooth. To reason about its time derivative, from Theorem II-1, we need to examine:

$$\dot{\tilde{V}} = \bigcap_{\xi \in \partial V(x)} \xi^T \cdot \begin{bmatrix} V_h & \underline{0} \\ \underline{0} & C \end{bmatrix} \cdot \mathcal{K}\begin{bmatrix} \mathbf{u_h} \\ \mathbf{u_{nh}} \end{bmatrix}$$

and since V is smooth,

$$\dot{\tilde{V}} = \nabla V^T \cdot \begin{bmatrix} V_h & \underline{0} \\ \underline{0} & C \end{bmatrix} \cdot \mathcal{K}\begin{bmatrix} \mathbf{u_h} \\ \mathbf{u_{nh}} \end{bmatrix}$$

Substituting the control law from Proposition 1 we get:

$$\dot{\tilde{V}} \subset -v_h - v_{nh_u} + v_{nh_w} \tag{II-1}$$

where

$$v_h = \delta_h$$

$$v_{nh_u} = \sum_{i=n+1}^{m}\left[|\nabla_{x_i,y_i} V \cdot \eta_i| \cdot \tau \frac{\check{v}_i^{max} \cdot Z_i}{a_2 + Z_i}\right] = \delta_{V_q}$$

where $\nabla_{x_i,y_i} V = [V_{x_i}, V_{y_i}]^T$ and $\eta_i = [\cos(\theta_i), \sin(\theta_i)]^T$. For v_{nh_w} we have

$$v_{nh_w} = \mathcal{K}\left[\sum_{i\in\mathcal{M}} [w_i \cdot V_{\theta_i} \cdot \check{w}_i^{max}]\right].$$

Then (II-1) becomes:

$$\begin{aligned}\dot{\tilde{V}} &\subset -v_h - v_{nh_u} + v_{nh_w} \\ &= -\delta_h - \delta_{V_q} + \mathcal{K}\left[\sum_{i\in\mathcal{M}}\left[w_i \cdot V_{\theta_i} \cdot \check{w}_i^{max}\right]\right] \\ &= -\delta_h - \delta_{V_q} + \mathcal{K}\left[\delta_{\theta_{nh}}(\rho)\right] \\ &\subseteq [\Delta_\rho, 0]\end{aligned}$$

since the switchings occur between negative values of $\Delta(\cdot)$ away of the target, while at the target $\rho = 2^z$ and $\Delta_\rho = 0$. The eventual set is closed due to the closure of operator $\mathcal{K}[\cdot]$.

Now let

$$S = \{\mathbf{x} : 0 \in \dot{\tilde{V}}(\mathbf{x})\}.$$

The set

$$S \supset M = \left\{\begin{array}{l}\mathbf{p} : u_{x_t} = u_{y_t} = \omega_t = \omega_i = u_i = 0, \\ \forall t \in \{1 \ldots n\}, \forall i \in \mathcal{M}\end{array}\right\}$$

is the largest invariant set. To see this consider that the only case that an invariant set would have a non-zero velocity would be a sliding surface forcing the system in a limit cycle. However this cannot happen since due to the analytic property of the MRNF we cannot have a zero curvature equipotential surface, so the sliding surfaces will be repulsive under the proposed control law (aligning θ_i with θ_{nh_i} makes the term $\delta_{V_q} \neq 0$ when not at the destination). From the proposed control law, it can be seen that $u_i = 0, \forall i \in \mathcal{M}$ only at the destination, and for all other configurations the controller provides a direction of movement and $\left\|u_{x_t}^2 + u_{y_t}^2\right\| > 0$ a.e. and vanishes at the origin. The set of initial conditions that lead the holonomic subsystem to saddle points is guaranteed to be of measure zero due to the Morse property (Property 3 in Definition 1) of MRNFs. According to LaSalle's invariance principle for non-smooth systems (Theorem II-2), the trajectories of the non-holonomic subsystem converge asymptotically to the largest invariant set, which is the destination configuration while the trajectories of the holonomic subsystem converge asymptotically to the destination configuration a.e. ■

B. Proof of Theorem 1.

Following a similar line of thought as in [45], we introduce the change of variables $z = \xi - \phi(\eta)$, $\mathbf{v} = \nu - \dot{\tilde{\phi}}$. System (10) can be written as

$$\begin{aligned}\dot{\eta} &= g(\eta)\phi(\eta) + g(\eta)z \\ \dot{z} &= \mathbf{v}.\end{aligned}$$

Since (Multi-Robot) Navigation Function $V(\eta)$ can be considered as a Lyapunov function candidate, let us consider the following:

$$V_a(\eta, \xi) \triangleq V(\eta) + \frac{1}{2}z^T z$$

which is a Lyapunov function candidate. Then every element $\delta \in \dot{\tilde{V}}_a$ satisfies

$$\delta \leq \mathbf{w} + \rho + z^T v$$

with

$$\mathbf{w} \in D \triangleq \bigcap_{\lambda \in \partial V(\eta)} \lambda^T \mathbf{F}(g(\eta)\phi(\eta)),$$

$$\rho \in \bigcap_{\lambda \in \partial V(\eta)} \lambda^T \mathbf{F}(\mathbf{g}(\eta)\mathbf{z})$$

and $v \in \mathbf{v}$. Since the control law

$$\xi = \phi(\eta)$$

with V a Lyapunov function stabilizes system (10a), the set D will only have non-positive elements, hence $\mathbf{w} \leq 0, \forall \mathbf{w} \in D$. With $\chi \in \dot{\tilde{\phi}}, \quad v = \nu - \chi$ after substitution, we get:

$$\delta \leq -z^T K_z z + \rho - V^o(\eta; g(\eta)[\xi - \phi(\eta)]) + z^T(\zeta - \chi) - z^T P z$$

where $P = diag(a|\xi| + b\mathbf{1}_m) \cdot diag(\mathbf{1}_m - |\xi|)^{-1}$. Now from the definition of the generalized gradient (see Definition I-2), for all $\rho \in \bigcap_{\lambda \in \partial V(\eta)} \lambda^T \mathbf{F}(g(\eta)z)$ we have that $\rho - V^o(\eta; g(\eta)z) \leq 0$. The term $\zeta - \chi$ is everywhere zero, since $\zeta = \chi$ when ϕ is differentiable. At points of non-differentiability, the Filippov solutions of (10b) are absolutely continuous hence the set $\mathbf{F}(g(\eta)\xi)$ is a singleton thus by the definition of generalized time derivative, $\dot{\tilde{\phi}}$ is empty, the minimum norm element of $\dot{\tilde{\phi}}$ is zero and $\zeta - \chi = 0$. To proceed with the proof we will need the following:

Lemma 1: The set $U = \{\xi : \|\xi\|_\infty \leq 1\}$ for the system (10b) under the control law (11) is positive invariant.

Proof: We need to show that at the boundary of U, the vector field generated by (11) is repulsive, i.e. points inwards the set U. To see this we note that although $\dot{\tilde{\phi}}$ may be locally unbounded, the selection of ζ ensures that (12) will always be bounded. Assume $B = \max_{\xi \in U}\{\|\sigma(\xi)\|_\infty\}$ and without losing generality, let B be the i'th element of σ. Now examining the i'th element of μ, we have $\mu_i = \frac{a|\xi_i| + b}{1 - |\xi_i|}(\xi_i - \phi_i(\eta))$. We need to show that there always exists a value of $|\xi_i|$ for $\xi \in H = U \bigcap \{\xi : \|\xi\|_\infty \geq \tau + \epsilon\}$ where $0 < \epsilon \leq 1 - \tau$ above which $|\mu_i| > B, \forall \phi_i(\eta)$. Since $|\phi_i(\eta)| \leq \tau$, we have $|\mu_i| \geq \frac{a|\xi_i| + b}{1 - |\xi_i|}(|\xi_i| - \tau)$ for $\xi \in H$. As can be seen for every finite value of B there is always a value of $\xi \in H$ such that $|\mu_i| > B$. Since $|\phi_i(\eta)| \leq \tau$ for every value of $\xi \in H$, μ_i will have the sign of ξ_i. Define the following subset of $H \supset H_B = \{\xi \in H : \|\mu\|_\infty > B\}$. Examining the i'th term of the control law (11), we have for $\xi_i = \|\xi\|_\infty$, $\xi \in H_B$ that $v_i \leq B - \mu_i < 0$ and for $\xi_i = -\|\xi\|_\infty, \xi \in H_B$ that $v_i \geq -B - \mu_i > 0$. Hence the set H_B and consequently the boundary of $U \supset H_B$ is repulsive. □

In view of Lemma 1, the term $z^T P z$ is non-negative hence every element of $\dot{\tilde{V}}_a$ is strictly negative almost everywhere except from the origin $(\eta, z) = (0, 0)$ and a set of measure zero of saddle points.

Now let $S = \{(\eta, \xi) : 0 \in \dot{\tilde{V}}_a(\mathbf{x})\}$ and $S \supset M = \left\{(\eta, \xi) : \dot{\eta} = \dot{\xi} = 0\right\}$ is an invariant set. For $\dot{\eta} = 0$ we have that $\xi = 0$ since $g(\eta) \neq 0$. Also since $\dot{\eta} = 0$ we have that $\zeta = 0$ and since $V^o(\eta; g(\eta)\phi(\eta)) \leq 0$, for $\dot{\xi} = 0$ it must be $\phi = 0$. By definition $\phi = 0$ only at the destination and at a set of measure zero of saddle points. Since (MR)NFs are regular functions (since they are smooth), we can apply LaSalle's invariance principle for non-smooth systems (Theorem II-2), according to which the trajectories of the system converge asymptotically to the destination configuration a.e. ■