Closed Loop Navigation for Multiple Holonomic Vehicles

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Abstract

We extend the navigation function methodology, established for single robot navigation, to the case of multiple robots. Appropriate expression of the robot potential functions guarantees global convergence. The derived closed form navigation function provides a robust navigation scheme, suitable for real time implementation. The collision avoidance and global convergence properties are verified through simulations.

1 Introduction

Navigation of mobile robots has been an area of great research interest in robotics. Most efforts have been focused at the case of a single robot navigating in an environment with obstacles. Recently, navigation for multiple mobile robots is gaining increasing attention. The basic motivation for this work comes from the field of micro robotics, where a team of autonomous micro robots must cooperate to achieve manipulation precision in the sub micron level. Multi robot navigation arises as a basic issue in this task.

There have been several attempts to treat the multi robot navigation problem. Methodologies of motion planning with multiple robots are divided into two large categories. The first category is *Centralized Planning*, which consists of planning the coordinated paths of multiple robots as a path in their composite configuration space: Barraquand, Langlois and Latombe [1] described an **off line** potential field based method to navigate disk shaped robots in narrow corridors. Tournassoud [16] proposed a variant of the potential field approach where motion coordination was expressed as a local optimization problem but global convergence could not be guaranteed since robots could reach a deadlock state where one robot was blocking the other. Barraquand and Latombe ([2],[3]) applied randomized path planner - an **off line** potential field based approach that uses random motions to escape local minima. The second category is *Decoupled Planning* ([8],[11]), in which paths for each robot are planned independently of other robots and then interactions between paths are considered: O'Donnel and Lozano-Perez [11] proposed *Path Coordination*, a decoupled planning approach based on a scheduling technique for dealing with limited resources. We are mainly interested in on line planners, so we will treat the first category of methodologies.

The multi robot navigation problem we treat in this paper can be stated as follows: "Derive a control law, which drives the robots from any initial configuration to the goal configuration avoiding collisions. The environment is assumed perfectly known and stationary, while each robot has global knowledge of the environment and the team configuration". Our basic idea is to use the gradient of a potential function to navigate a team of robots, while each robot acts as a potential obstacle to the others.

The rest of the paper is organized as follows: Section 2 outlines the concept of navigation functions. Section 3 introduces the new terminology and the mathematical tools required for the analysis. A proof of correctness is provided in this section.

2 Navigation Functions

Navigation functions are real valued maps realized through cost functions, whose negated gradient field is attractive towards the goal configuration and repulsive wrt obstacles. It has been shown (Koditschek and Rimon [6]) that strict global navigation (i.e. with a globally attracting equilibrium state) is not possible and a smooth vector field on any sphere world, which has a unique attractor, must have at least as many saddles as obstacles. Our assumption that we have spherical robots and spherical obstacles does not constrain the generality of this work since it has been proven [6] that navigation properties are invariant under diffeomorphisms. Methods for constructing analytic diffeomorphisms are discussed in ([13],[12]) for point robots and in [14] for rigid body robots.

Let us assume the following situation: We have m mobile robots, and their workspace $W \subset R^2$. Each robot R_i, $i = 1 \dots m$ occupies a disk in the workspace: $R_i = \{q \in R^2 : \|q - q_i\| \le r_i\}$ where $q_i \in R^2$ is the center of the disk and r_i is the radius of the robot. The configuration of each robot is represented by q_i and the configuration space C is spanned by $q = \left[q_1^T \dots q_m^T\right]^T$. A navigation function can be defined as follows:

Definition 1. *Let $F \subset R^n$ be a compact connected analytic manifold with boundary. A map $\varphi : F \to [0,1]$ is a navigation function if:*

1. *Is analytic on F,*
2. *It has only one minimum at $q_d \in \overset{o}{F}$,*
3. *Its Hessian at all critical points (zero gradient vector field) is full rank and*
4. $\lim\limits_{q \to \partial F} \varphi(q) = 1$

Essentially, the sought control law will be $\dot{q} = u$ where

$$u = -K \cdot \nabla\varphi\,(q)$$

, where K is a gain. We will prove that the class of navigation functions introduced in [6] for single robot navigation, if properly extended, can be used in the multi robot case. We consider the class of navigation functions

$$\varphi = \sigma_d \circ \sigma \circ \hat{\varphi} = \left(\frac{\gamma}{\gamma + G}\right)^{1/k} \tag{1}$$

that is a composition of $\sigma_d = x^{\frac{1}{k}}$, $\sigma = \frac{x}{1+x}$ and

$$\hat{\varphi} = \frac{\gamma}{G} \tag{2}$$

the cost function, for which $\gamma^{-1}(0)$ denotes the desirable set and $G^{-1}(0)$ the set, which we want to avoid. A suitable choice is $\gamma = \gamma_d^k$, where $\gamma_d = \|q - q_d\|^2$ is the squared metric of the current configuration q from the destination q_d. The following theorem will help us on deriving results for the function φ by examining the simpler function $\hat{\varphi}$:

Theorem 1 ([6]). *Let $I_1, I_2 \subseteq R$ be intervals, $\hat{\varphi} : F \to I_1$ and $\sigma : I_1 \to I_2$ be analytic. Define the composition $\varphi : F \to I_2$, to be $\varphi = \sigma \circ \hat{\varphi}$. If σ is monotonically increasing on I_1, then the set of critical points of $\hat{\varphi}$ and φ coincide and the (Morse) index of each critical point is identical.*

3 Mathematical Tools - Terminology

The robot proximity functions, a measure for the distance between two robots i and j, are defined by: $\beta_{i,j}(q) = q^T D_{ij} q - (r_i + r_j)^2$, where r_i is the radius of the i'th robot and

$$D_{ij} = \begin{bmatrix} & & O_{2(i-1)x2m} & & \\ O_{2x2(i-1)} & I_{2x2} & O_{2x2(j-i-1)} & -I_{2x2} & O_{2x2(m-j)} \\ & & O_{2(j-i-1)x2m} & & \\ O_{2x2(i-1)} & -I_{2x2} & O_{2x2(j-i-1)} & I_{2x2} & O_{2x2(m-j)} \\ & & O_{2(m-j)x2m} & & \end{bmatrix}$$

Some useful properties are: $D_{ij} = D_{ij}^T$, $D_{ij} = D_{ji}$, $D_{ij}^k = 2^{k-1} D_{ij}$. The gradient and hessian of $\beta_{i,j}$ are $\nabla \beta_{i,j}(q) = 2D_{ij} \cdot q$, because $\nabla q = \nabla q^T = I$ and $\nabla^2 \beta_{i,j}(q) = 2D_{ij}$. We will use the term '**relation**' to describe the possible collision schemes that can be defined in a multi robot - obstacles scene. The '**set of relations**' between the members of a set can be defined as the set of all possible collision schemes between the members. A **binary relation** is a relation between two robots. Any relation can be expressed as a set of binary relations. A '**relation tree**' is the set of robots-obstacles that form a linked team. Each *relation* may consist of more than one tree (figure 1). We will call the number of binary relations in a relation, the '**relation level**'.

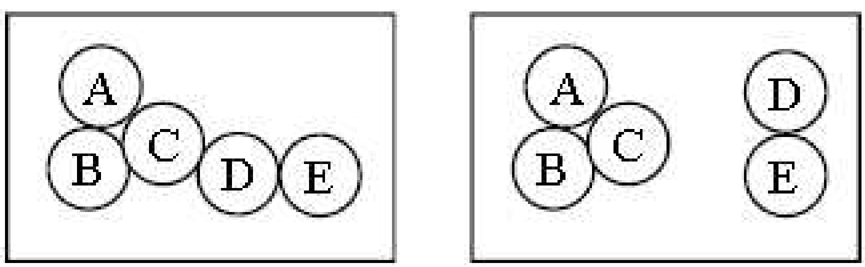

Figure 1 *: (a) One – tree relation, (b) Two tree relation*

A **relation proximity function (RPF)** provides a measure of the distance between the robots involved in a relation. Each relation has it's own *RPF*. An *RPF* assumes the value of zero whenever the related robots collide and increases wrt the distance of the related robots:

$$b_R = q^T \cdot P_R \cdot q - \sum_{\{i,j\} \in R} (r_i + r_j)^2 \tag{3}$$

where R is the set of binary relations (e.g. for the relation in figure (1.b) $R = \{\{A, B\}, \{A, C\}, \{B, C\}, \{D, E\}\}$) and $P_R = \sum\limits_{\{i,j\} \in R} D_{i,j}$ is the **rela-**

tion matrix of *RPF*. The gradient and Hessian of the *RPF* are: $\nabla b_R = 2P_R \cdot q$ and $\nabla^2 b_R = 2P_R$.

A **Relation Verification Function (RVF)** is defined by:

$$g_{R_j}\left(b_{R_j}, B_{R_j^C}\right) = b_{R_j} + \lambda \cdot b_{R_j} \Big/ \left(b_{R_j} + B_{R_j^C}^{1/h}\right) \tag{4}$$

where $\lambda, h > 0$, R_j^C is the complementary to R_j set of *relations* in the same level, j is an index number defining the relation in the level and $B_{R_j^C} = \prod\limits_{k \in R_j^C} b_k$. An *RVF* is zero if a relation holds while no other relation from the same level holds and has the properties: (a) $\lim\limits_{x \to 0} \lim\limits_{y \to 0} g_x(x, y) = \lambda$, (b) $\lim\limits_{y \to 0} \lim\limits_{x \to 0} g_x(x, y) = 0$.

Based on the above properties, in a robot proximity situation, one can verify that: if $\left(g_{R_j}\right)_k = 0$ at some *level* k then $\left(g_{R_i}\right)_h \neq 0$ for any *level* h and $i \neq j$ in level k . It should be noted hereby that since in the highest relation level only one relation exists, there will be no complementary relations and the RVF will be identical to the RPF e.g. $\lambda = 0$ for this relation.

The gradient and the Hessian of $g_x(x, y)$ are given below:

$$\nabla g\left(b, \tilde{b}\right) = \nabla b \cdot \left(1 + \frac{\lambda}{b + \tilde{b}^{1/h}}\right) - \frac{b \cdot \lambda}{\left(b + \tilde{b}^{1/h}\right)^2}\left(\nabla b + \nabla \tilde{b}^{1/h}\right) \tag{5}$$

$$\nabla^2 g\left(b, \tilde{b}\right) = \left(1 + \tfrac{\lambda}{b + \tilde{b}^{1/h}}\right) \cdot \nabla^2 b - \tfrac{2\lambda}{\left(b + \tilde{b}^{1/h}\right)^2} \nabla b \cdot \left(\nabla b + \nabla \tilde{b}^{1/h}\right)^T + \\ b \cdot \lambda \left(\tfrac{2}{\left(b + \tilde{b}^{1/h}\right)^3}\left(\nabla b + \nabla \tilde{b}^{1/h}\right)\left(\nabla b + \nabla \tilde{b}^{1/h}\right)^T - \tfrac{1}{\left(b + \tilde{b}^{1/h}\right)^2}\left(\nabla^2 b + \nabla^2 \tilde{b}^{1/h}\right)\right) \tag{6}$$

We can now define $G = \prod\limits_{L=1}^{n_L} \prod\limits_{j=1}^{n_{R,L}} \left(g_{R_j}\right)_L$, with n_L the number of *levels* and n_R, L the number of *relations* in *level* L . Figure (2) demonstrates several types of relations of a four – member team.

3.1 Proof of correctness

Let $\varepsilon > 0$. Define $B_i^l(\varepsilon) = \{q : 0 < (g_{R_i}(q))_l < \varepsilon\}$. We can then discriminate the following topologies:

1. The destination point q_d
2. The free space boundary: $\partial F(q) = G^{-1}(\delta)$, $\delta \to 0$

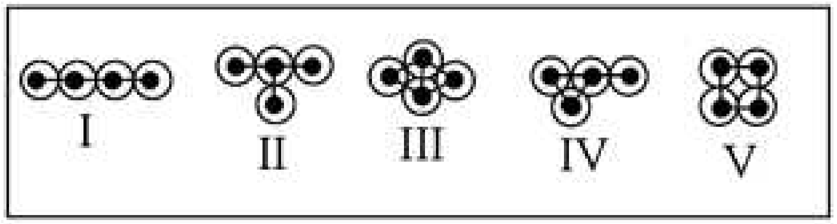

Figure 2 *: I, II are level 3; IV, V are level 4 and III is a level 5 relation*

3. The robot/obstacle proximity set: $F_0(\varepsilon) = \bigcup_{L=1}^{n_L} \bigcup_{i=1}^{n_{R,L}} B_i^L(\varepsilon) - \{q_d\}$, with n_L and n_R, L as defined above.

4. The robot/obstacle distant set: $F_1(\varepsilon) = F - (\{q_d\} \bigcup F_0(\varepsilon))$

Proposition 1. *The destination point q_d is a non – degenerate local minimum of φ.*

Proof. Similar to this found in [6]. From eq. (1), we have:

$$\nabla\varphi(q_d) = \frac{1}{\left(\gamma_d^k + G\right)^{2/k}} \left(\left(\gamma_d^k + G\right)^{1/k} \nabla\gamma_d - \gamma_d \nabla\left(\gamma_d^k + G\right)^{1/k}\right) = 0$$

since at q_d both γ_d and $\nabla\gamma_d$ are zero. The Hessian at a critical point is:

$$\nabla^2\varphi = \frac{1}{\left(\gamma_d^k + G\right)^{2/k}} \left(\left(\gamma_d^k + G\right)^{1/k} \nabla^2\gamma_d - \gamma_d \nabla^2\left(\gamma_d^k + G\right)^{1/k}\right)$$

but at qd, $\nabla^2\gamma_d = 2I$ and the Hessian reduces to:

$$\left(\nabla^2\varphi\right)(q_d) = 2G^{-1/k} I$$

which is non – degenerate. □

Proposition 2. *All the critical points are in the interior of the free space.*

Proof. Let q_0 be a point on ∂F and suppose that $\left(g_{R_j}\right)_\kappa(q_0) = 0$ for the relation j of level k. Then $(g_{R_i})_h(q_0) > 0$, for any *level* h and $i \neq j$ in level k, because only one *RVF* can hold at a time. Then at q_0:

$$\nabla\varphi(q_0) = \frac{1}{\left(\gamma_d^k + G\right)^{\frac{2}{k}}} \left(\left(\gamma_d^k + G\right)^{\frac{1}{k}} \nabla\gamma_d - \gamma_d \nabla\left(\gamma_d^k + G\right)^{\frac{1}{k}}\right)_{|q_0} =$$

$$= -\frac{1}{k}\gamma_d^{-k}\left(\prod_{L=1}^{n_L}\prod_{\substack{i(L)=1\\ i(k)\neq j}}^{n_{R,L}}\left(g_{R_i}\right)_L\right)\cdot\nabla\left(g_{R_j}\right)_k \neq 0$$

□

Proposition 3. *For every $\varepsilon > 0$, there exists a positive integer $\mathrm{N}(\varepsilon)$ such that if $k > \mathrm{N}(\varepsilon)$ then there are no critical points of $\hat{\varphi}$ in $F_1(\varepsilon)$.*

Proof. Similar to this found in [6]. From eq. (2) it follows:

$$\nabla\hat{\varphi} = \frac{1}{G^2}\left(Gk\gamma_d^{k-1}\nabla\gamma_d - \gamma_d^k\nabla G\right)$$

At a critical point it will be:

$$\gamma_d\nabla G = Gk\nabla\gamma_d$$

Taking the magnitude of both sides:

$$2\kappa G = \sqrt{\gamma_d}\left\|\nabla G\right\|$$

because $\left\|\nabla\gamma_d\right\| = 2\sqrt{\gamma_d}$. A sufficient condition for the above equality not to hold is:

$$\kappa > \frac{1}{2}\frac{\sqrt{\gamma_d}\left\|\nabla G\right\|}{G}$$

for all

$$q \in F_1(\varepsilon)$$

An upper bound for the right side of the inequality can be derived, provided that the workspace (or configuration space) C is bounded and is given by:

$$\frac{1}{2}\frac{\sqrt{\gamma_d}\left\|\nabla G\right\|}{G} < \frac{1}{2}\frac{1}{\varepsilon}\max_C\left\{\sqrt{\gamma_d}\right\}\sum_{L=1}^{n_L}\sum_{j=1}^{n_{R,L}}\max_C\left\{\left\|\nabla\left(g_{R_j}\right)_L\right\|\right\} \triangleq N(\varepsilon)$$

since $\left(g_{R_j}\right)_L > \varepsilon, j \in \{0..n_{R,L}\}, L \in \{0..n_L\}$. □

Hence the set away from the obstacles is 'cleaned' from critical points. The workspace can be bounded with several obstacles prohibiting the motion of robots beyond them or by defining a world obstacle in the sense of robot proximity function: $\beta_{w,i} = (-1)\left(q_i^T q_i - (r_w - r_i)^2\right)$ where the index i refers to the robot and the index w refers to the world obstacle.

Proposition 4. *There exists an $\varepsilon_0 > 0$, such that $\hat{\varphi}$ has no local minimum in $F_0(\varepsilon)$, as long as $\varepsilon < \varepsilon_0$.*

Proof. If $q \in F_0(\varepsilon) \cap C_{\hat{\varphi}}$, where $C_{\hat{\varphi}}$ is the set of critical points, then $q \in B_i^L(\varepsilon)$ for at least one set $\{L, i\}$, $i \in \{1..n_{R,L}\}$, $L \in \{1 \ldots n_L\}$ with n_L the number of *levels* and $n_{R,L}$ the number of *relations* in *level* L . We will use a unit vector as a test direction to demonstrate that $(\nabla^2 \hat{\varphi})(q)$ has at least one negative eigenvalue. At a critical point,

$$(\nabla \hat{\varphi})(q) = \frac{1}{G^2}\left(k \cdot G \cdot \gamma_d^{k-1} \cdot \nabla \gamma_d - \gamma_d^k \cdot \nabla G\right) = 0$$

Hence

$$\gamma_d \nabla G = G k \nabla \gamma_d \tag{7}$$

The Hessian at a critical point is:

$$\left(\nabla^2 \hat{\varphi}\right)(q) = \frac{1}{G^2}\left(G \cdot \nabla^2 \gamma_d^k - \gamma_d^k \cdot \nabla^2 G\right)$$

and expanding:

$$\left(\nabla^2 \hat{\varphi}\right)(q) = \frac{\gamma_d^{k-2}}{G^2}\left(kG\left[\gamma_d \cdot \nabla^2 \gamma_d + (k-1)\nabla \gamma_d \nabla \gamma_d^T\right] - \gamma_d^2 \cdot \nabla^2 G\right) \tag{8}$$

Taking the outer product of both sides of eq. (7), we get:

$$(Gk)^2 \nabla \gamma_d \nabla \gamma_d^T = \gamma_d^2 \nabla G \cdot \nabla G^T \tag{9}$$

Substituting eq. (9) in eq. (8), we get:

$$\left(\nabla^2 \hat{\varphi}\right)(q) = \frac{\gamma_d^{k-1}}{G^2}\left(kG \cdot \nabla^2 \gamma_d + \left(1 - \frac{1}{k}\right)\frac{\gamma_d}{G}\nabla G \cdot \nabla G^T - \gamma_d \cdot \nabla^2 G\right) \tag{10}$$

We choose the test vector to be: $\hat{u} = P_{R_i} \cdot q^\perp / \left\| P_{R_i} \cdot q^\perp \right\|$ where P_{R_i} is the *relation matrix* of b_{R_i} and $\left(q^\perp\right)^T = \left[\begin{array}{ccc} q_1^\perp & \ldots & q_m^\perp \end{array}\right]$. With $\nabla^2 \gamma_d = 2I$ we form the quadratic form:

$$\frac{G^2}{\gamma_d^{k-1}} \hat{u}^{\mathrm{T}} \left(\nabla^2 \hat{\varphi}\right)(q)\, \hat{u} = 2kG + \left(1 - \frac{1}{k}\right)\frac{\gamma_d}{G}\hat{u}^{\mathrm{T}} \cdot \nabla G \cdot \nabla G^T \cdot \hat{u} - \gamma_d \cdot \hat{u}^{\mathrm{T}} \cdot \nabla^2 G \cdot \hat{u} \tag{11}$$

Taking the inner product of u and ∇b_{R_i} we have:

$$\left\langle \left(2P_{R_i} \cdot q\right), \left(P_{R_i} \cdot q^\perp\right)\right\rangle = 2q^T P_{R_i}^T P_{R_i} q^\perp$$

According to proposition (A-1) found in the appendix, the product $P_{R_i}^T P_{R_i}$, is a linear combination of the matrices $D_{i,j}$, with $\{i, j\} \in \mathbf{P^2}_{R_i}$ where $\mathbf{P^2}$ is defined in the aforementioned proposition. Hence we can write:

$$P_{R_i}^T \cdot P_{R_i} = \sum_{\{i,j\} \in \mathbf{P^2}_{R_i}} a_{ij} D_{ij}$$

with $a_{i,j}$ integer constants (see prop. (A-1)). So:

$$
\begin{aligned}
& q^T \cdot P_{R_i}^T \cdot P_{R_i} \cdot q^\perp = q^T \left(\sum_{\{i,j\} \in \mathbf{P^2}_{R_i}} a_{ij} D_{ij} \right) q^\perp = \\
& = \sum_{\{i,j\} \in \mathbf{P^2}_{R_i}} a_{ij} q^T D_{ij} q^\perp = \sum_{\{i,j\} \in \mathbf{P^2}_{R_i}} \left(a_{ij} q^T \left[\begin{array}{cc} O_{1x(2i-1)} & (q_i - q_j)_{1x2}^T \ldots \end{array} \right. \right. \\
& \qquad \left. \left. \begin{array}{ccc} \ldots O_{1x2(j-i-1)} & (q_j - q_i)_{1x2}^T & O_{1x2(m-j)} \end{array} \right] \cdot q^\perp \right) = \\
& = \sum_{\{i,j\} \in \mathbf{P^2}_{R_i}} a_{ij} \left((q_i - q_j)^T \cdot q_i^\perp + (q_j - q_i)^T \cdot q_j^\perp \right) = \\
& = \sum_{\{i,j\} \in \mathbf{P^2}_{R_i}} a_{ij} (q_i - q_j)^T \cdot \left(q_i^\perp - q_j^\perp \right) = \sum_{\{i,j\} \in \mathbf{P^2}_{R_i}} a_{ij} (q_i - q_j)^T \cdot (q_i - q_j)^\perp = 0
\end{aligned}
$$

Hence $\hat{u} \perp \nabla b_{R_i}$. In the following analysis we will use the subscript 'i' instead of 'R_i' to simplify notation.

Expanding the term $\hat{u}^{\mathrm{T}} \cdot \nabla G \cdot \nabla G^T \cdot \hat{u}$ in eq. (11), we get:

$$
\hat{u}^{\mathrm{T}} \cdot \nabla G \cdot \nabla G^T \cdot \hat{u} = \hat{u}^{\mathrm{T}} \cdot \left((g_i \cdot \nabla \bar{g}_i + \bar{g}_i \cdot \nabla g_i) \cdot \left(g_i \cdot \nabla \bar{g}_i^T + \bar{g}_i \cdot \nabla g_i^T \right) \right) \cdot \hat{u} \quad (12)
$$

where $G = g_i \cdot \bar{g}_i$. But taking the inner product of $\hat{u}$ with eq. (5), we get:

$$
\hat{u}^{\mathrm{T}} \cdot \nabla g_i = -b_i \frac{\lambda}{\left(b_i + \tilde{b}_i^{1/h} \right)^2} \hat{u}^{\mathrm{T}} \cdot \nabla \tilde{b}_i^{1/h}
$$

and eq. (12) simplifies to:

$$
\begin{aligned}
& \hat{u}^{\mathrm{T}} \cdot \nabla G \cdot \nabla G^T \cdot \hat{u} = \\
& = g_i^2 \hat{u}^{\mathrm{T}} \cdot \nabla \bar{g}_i \cdot \nabla \bar{g}_i^T \cdot \hat{u} + \tfrac{\lambda^2 \bar{g}_i^2 b_i^2}{\left(b_i + \tilde{b}_i^{1/h} \right)^4} \hat{u}^{\mathrm{T}} \cdot \nabla \tilde{b}_i^{1/h} \cdot (\nabla \tilde{b}_i^{1/h})^T \hat{u} - \ldots \\
& \cdots - \tfrac{2 g_i \bar{g}_i b_i \lambda}{\left(b_i + \tilde{b}_i^{1/h} \right)^2} \hat{u}^{\mathrm{T}} \cdot \nabla \tilde{b}_i^{1/h} \cdot \nabla \bar{g}_i^T \cdot \hat{u}
\end{aligned}
$$

Using $g_i = c_i \cdot b_i$, where $c_i = \left(1 + \lambda \middle/ \left(b_i + \tilde{b}_i^{1/h} \right) \right)$ we get:

$$
\begin{aligned}
& \hat{u}^{\mathrm{T}} \cdot \nabla G \cdot \nabla G^T \cdot \hat{u} = \\
& = g_i^2 \hat{u}^{\mathrm{T}} \cdot \nabla \bar{g}_i \cdot \nabla \bar{g}_i^T \cdot \hat{u} + \tfrac{\lambda^2 G^2}{c_i^2 \left(b_i + \tilde{b}_i^{1/h} \right)^4} \hat{u}^{\mathrm{T}} \cdot \nabla \tilde{b}_i^{1/h} \cdot (\nabla \tilde{b}_i^{1/h})^T \hat{u} - \ldots \\
& \cdots - \tfrac{2 G \cdot g_i \lambda}{c_i \cdot \left(b_i + \tilde{b}_i^{1/h} \right)^2} \hat{u}^{\mathrm{T}} \cdot \nabla \tilde{b}_i^{1/h} \cdot \nabla \bar{g}_i^T \cdot \hat{u}
\end{aligned}
$$

Hence

$$
\left(1 - \frac{1}{k} \right) \frac{\gamma_d}{G} \hat{u}^{\mathrm{T}} \cdot \nabla G \cdot \nabla G^T \cdot \hat{u} = g_i \gamma_d \cdot \eta_i
$$

where

$$\eta_i = \left(\frac{\hat{u}^{\mathrm{T}} \cdot \nabla \bar{g}_i \cdot \nabla \bar{g}_i^T \cdot \hat{u}}{\bar{g}_i} + \frac{\lambda^2 \cdot \bar{g}_i \cdot \hat{u}^{\mathrm{T}} \cdot \nabla \left(\tilde{b}_i^{1/h} \right) \cdot \nabla \left(\tilde{b}_i^{1/h} \right)^T}{c_i^2 \left(b_{R_i} + \tilde{b}_i^{1/h} \right)^4} \dots \right.$$

$$\left. \dots - \frac{2\lambda}{c_i \cdot \left(b_{R_i} + \tilde{b}_i^{1/h} \right)^2} \hat{u}^{\mathrm{T}} \cdot \nabla \left(\tilde{b}_i^{1/h} \right) \cdot \nabla \bar{g}_i^T \cdot \hat{u} \right) \left(1 - \frac{1}{k} \right)$$

Expanding the term $\hat{u}^{\mathrm{T}} \cdot \nabla^2 G \cdot \hat{u}$ of eq. (11), we get:

$$\begin{aligned}
&\hat{u}^{\mathrm{T}} \cdot \nabla^2 G \cdot \hat{u} = \hat{u}^{\mathrm{T}} \cdot \left(\nabla^2 \left(g_i \cdot \bar{g}_i \right) \right) \cdot \hat{u} = \\
&= \hat{u}^{\mathrm{T}} \cdot \left(g_i \cdot \nabla^2 \bar{g}_i + 2 \left(\nabla g_i \cdot \nabla \bar{g}_i^{\mathrm{T}} \right)_s + \bar{g}_i \cdot \nabla^2 g_i \right) \cdot \hat{u} = \\
&= g_i \cdot \hat{u}^{\mathrm{T}} \cdot \nabla^2 \bar{g}_i \cdot \hat{u} + \bar{g}_i \cdot \hat{u}^{\mathrm{T}} \cdot \nabla^2 g_i \cdot \hat{u} - 2 \frac{b_i \lambda}{\left(b_i + \tilde{b}_i^{1/h} \right)^2} \hat{u}^{\mathrm{T}} \cdot \nabla \tilde{b}_i^{1/h} \cdot \nabla \bar{g}_i \cdot \hat{u}
\end{aligned}$$

but from eq. (6) :

$$\hat{u}^{\mathrm{T}} \cdot \nabla^2 g_i \cdot \hat{u} = \left(1 + \frac{\lambda}{b_i + \tilde{b}_i^{1/h}} \right) \cdot \hat{u}^{\mathrm{T}} \cdot \nabla^2 b_i \cdot \hat{u} + b_i \cdot \hat{u}^{\mathrm{T}} \cdot A_i \cdot \hat{u}$$

where

$$A_i = \lambda \left(2 \frac{\left(\nabla b_i + \nabla \tilde{b}_i^{1/h} \right) \left(\nabla b_i + \nabla \tilde{b}_i^{1/h} \right)^T}{\left(b_i + \tilde{b}_i^{1/h} \right)^3} - \frac{\left(\nabla^2 b_i + \nabla^2 \tilde{b}_i^{1/h} \right)}{\left(b_i + \tilde{b}_i^{1/h} \right)^2} \right)$$

Using $\nabla^2 b_k (q) = 2 \sum_{\{i,j\} \in R_i} D_{ij} = 2P_i$, the term $\hat{u}^{\mathrm{T}} \cdot \nabla^2 b_i \cdot \hat{u}$ becomes:

$$\hat{u}^{\mathrm{T}} \cdot \nabla^2 b_i \cdot \hat{u} = \frac{1}{\| 2P \cdot q^{\perp} \|^2} \left(2 \left(q^{\perp} \right)^T \cdot P \right) \cdot 2P \cdot \left(2P \cdot q^{\perp} \right) = 2 \frac{\left(q^{\perp} \right)^T \cdot P \cdot P \cdot P \cdot q^{\perp}}{\left(q^{\perp} \right)^T \cdot P \cdot P \cdot q^{\perp}}$$

To further simplify notation the index term 'i' referring to the set R_i is omitted when possible. From proposition (A-1) we have that $P.P = 2P + R$, where:

$$R = \sum_{i=1}^{k-1} \sum_{j=i+1}^{k} \Delta_{p_i, q_i, p_j, q_j}$$

is defined in (A-1). Hence

$$\frac{\left(q^{\perp} \right)^T \cdot P \cdot P \cdot P \cdot q^{\perp}}{\left(q^{\perp} \right)^T \cdot P \cdot P \cdot q^{\perp}} = \frac{\left(q^{\perp} \right)^T \cdot P \left(2P + R \right) \cdot q^{\perp}}{\left(q^{\perp} \right)^T \cdot P \cdot P \cdot q^{\perp}} = 2 + \frac{\left(q^{\perp} \right)^T \cdot P \cdot R \cdot q^{\perp}}{\left(q^{\perp} \right)^T \cdot P \cdot P \cdot q^{\perp}}$$

Using results from propositions (A-2) and (A-3) found in the Appendix, we have:

$$P \cdot P = 2P + R$$

$$P \cdot R = \sum_{i=1}^{T} \sum_{j=1}^{n_{tree_t}} \left(\left(c_{j,i}^2 - 1\right) \sum_{m=1}^{c_{j,i}} D_{p_j,q_m} - c_{j,i} \sum_{m=1}^{c_{j,i}-1} \sum_{n=m+1}^{c_{j,i}} D_{p_m,q_n} \right)$$

and using the notation introduced in (A-3):

$$R = \sum_{i=1}^{T} \sum_{j=1}^{n_{tree_t}} \left((c_{j,i} - 1) \sum_{m=1}^{c_{j,i}} D_{p_j,q_m} - \sum_{m=1}^{c_{j,i}-1} \sum_{n=m+1}^{c_{j,i}} D_{p_m,q_n} \right)$$
$$P = \sum_{i=1}^{T} \sum_{j=1}^{n_{tree_t}} \sum_{m=1}^{c_{j,i}} D_{p_j,q_m}$$

we get:

$$\frac{\left(q^{\perp}\right)^T \cdot P \cdot P \cdot P \cdot q^{\perp}}{\left(q^{\perp}\right)^T \cdot P \cdot P \cdot q^{\perp}} = 2 + \frac{\left(q^{\perp}\right)^T \cdot \sum_{i=1}^{T} \sum_{j=1}^{n_{tree_t}} \left(\left(c_{j,i}^2 - 1\right) \sum_{m=1}^{c_{j,i}} D_{p_j,q_m} - c_{j,i} \sum_{m=1}^{c_{j,i}-1} \sum_{n=m+1}^{c_{j,i}} D_{p_m,q_n} \right) \cdot q^{\perp}}{\left(q^{\perp}\right)^T \cdot \sum_{i=1}^{T} \sum_{j=1}^{n_{tree_t}} \left((c_{j,i} + 1) \sum_{m=1}^{c_{j,i}} D_{p_j,q_m} - \sum_{m=1}^{c_{j,i}-1} \sum_{n=m+1}^{c_{j,i}} D_{p_m,q_n} \right) \cdot q^{\perp}} =$$
$$= 2 + \frac{\sum_{i=1}^{T} \sum_{j=1}^{n_{tree_t}} \left(\left(c_{j,i}^2 - 1\right) \sum_{m=1}^{c_{j,i}} \left(q^{\perp}\right)^T \cdot D_{p_j,q_m} \cdot q^{\perp} - c_{j,i} \sum_{m=1}^{c_{j,i}-1} \sum_{n=m+1}^{c_{j,i}} \left(q^{\perp}\right)^T \cdot D_{p_m,q_n} \cdot q^{\perp} \right)}{\sum_{i=1}^{T} \sum_{j=1}^{n_{tree_t}} \left((c_{j,i} + 1) \sum_{m=1}^{c_{j,i}} \left(q^{\perp}\right)^T \cdot D_{p_j,q_m} \cdot q^{\perp} - \sum_{m=1}^{c_{j,i}-1} \sum_{n=m+1}^{c_{j,i}} \left(q^{\perp}\right)^T \cdot D_{p_m,q_n} \cdot q^{\perp} \right)} \tag{13}$$

We will now examine for each node the following terms:

$$\sum_{m=1}^{c_{j,i}} \left(q^{\perp}\right)^T \cdot D_{p_j,q_m} \cdot q^{\perp}$$

and

$$\sum_{m=1}^{c_{j,i}-1} \sum_{n=m+1}^{c_{j,i}} \left(q^{\perp}\right)^T \cdot D_{p_m,q_n} \cdot q^{\perp}$$

The transformation from q to $q^{\perp}$ is a pure 90-degree rotation, which conserves the angles and distances. So

$$\sum_{m=1}^{c_{j,i}} \left(q^{\perp}\right)^T \cdot D_{p_j,q_m} \cdot q^{\perp} = \sum_{m=1}^{c_{j,i}} q^T \cdot D_{p_j,q_m} \cdot q$$

and

$$\sum_{m=1}^{c_{j,i}-1} \sum_{n=m+1}^{c_{j,i}} \left(q^{\perp}\right)^T \cdot D_{p_m,q_n} \cdot q^{\perp} = \sum_{m=1}^{c_{j,i}-1} \sum_{n=m+1}^{c_{j,i}} q^T \cdot D_{p_m,q_n} \cdot q$$

Assuming the distances of the robots from the node robot to be negligible, we have:

$$\sum_{m=1}^{c_{j,i}} \left(q^{\perp}\right)^{T} \cdot D_{p_j,q_m} \cdot q^{\perp} = c_{j,i} \cdot r_j + \sum_{m=1}^{c_{j,i}} r_m \tag{14}$$

$$(c_{j,i}-1)\cdot\sum_{m=1}^{c_{j,i}} r_m < \sum_{m=1}^{c_{j,i}-1} \sum_{n=m+1}^{c_{j,i}} \left(q^{\perp}\right)^{T} \cdot D_{p_m,q_n} \cdot q^{\perp} < \begin{pmatrix} c_{j,i} \\ 2 \end{pmatrix} \cdot 2 \cdot r_j + (c_{j,i}-1)\cdot\sum_{m=1}^{c_{j,i}} r_m \tag{15}$$

The first relation (eq. (14)) is obvious. The second relation (eq. (15)) is bounded between two extreme situations: The lower bound is when all the non node robots are very close to each other. The upper bound is when all the non node robots are as far as they can be from each other. The term in the numerator in eq. (13), then becomes (lower bound):

$$\begin{aligned}&\left(c_{j,i}^2-1\right)\left(c_{j,i}\cdot r_j + \sum_{m=1}^{c_{j,i}} r_m\right) - c_{j,i}\left(\begin{pmatrix} c_{j,i} \\ 2 \end{pmatrix} \cdot 2 \cdot r_j + (c_{j,i}-1)\cdot \sum_{m=1}^{c_{j,i}} r_m\right) = \\ &= (c_{j,i}-1)\left(c_{j,i}\cdot r + \sum_{m=1}^{c_{j,i}} r_m\right)\end{aligned}$$

And the term in the denominator becomes (lower bound):

$$\begin{aligned}&(c_{j,i}+1)\left(c_{j,i}\cdot r_j + \sum_{m=1}^{c_{j,i}} r_m\right) - \left(\begin{pmatrix} c_{j,i} \\ 2 \end{pmatrix} \cdot 2 \cdot r_j + (c_{j,i}-1)\cdot \sum_{m=1}^{c_{j,i}} r_m\right) = \\ &= 2\left(c_{j,i}\cdot r_j + \sum_{m=1}^{c_{j,i}} r_m\right)\end{aligned}$$

Then from eq. (13) we get:

$$\frac{\left(q^{\perp}\right)^{T} \cdot P \cdot P \cdot P \cdot q^{\perp}}{\left(q^{\perp}\right)^{T} \cdot P \cdot P \cdot q^{\perp}} > 2 + \frac{\sum_{i=1}^{T} \sum_{j=1}^{n_{tree_t}} \left((c_{j,i}-1)\left(c_{j,i}\cdot r + \sum_{m=1}^{c_{j,i}} r_m\right)\right)}{\sum_{i=1}^{T} \sum_{j=1}^{n_{tree_t}} \left(2\left(c_{j,i}\cdot r_j + \sum_{m=1}^{c_{j,i}} r_m\right)\right)}$$

And since $c_{j,i} \geq 1$, we have that:

$$\frac{\left(q^{\perp}\right)^{T} \cdot P \cdot P \cdot P \cdot q^{\perp}}{\left(q^{\perp}\right)^{T} \cdot P \cdot P \cdot q^{\perp}} = v_i$$

where $v_i > 2$. So

$$\hat{u}^{\mathrm{T}} \cdot \nabla^2 g_i \cdot \hat{u} = v_i \cdot c_i + b_i \cdot \hat{u}^{\mathrm{T}} \cdot A_i \cdot \hat{u}$$

where

$$c_i = \left(1 + \frac{\lambda}{b_i + \tilde{b}_i^{1/h}}\right)$$

and

$$\begin{aligned}&\hat{u}^{\mathrm{T}} \cdot \nabla^2 G \cdot \hat{u} = \\ &= g_i \cdot \hat{u}^{\mathrm{T}} \cdot \nabla^2 \bar{g}_i \cdot \hat{u} + \bar{g}_i \cdot \left(v_i \cdot c_i + b_i \cdot \hat{u}^{\mathrm{T}} \cdot A_i \cdot \hat{u}\right) - 2\tfrac{b_i \lambda}{\left(b_i + \tilde{b}_i^{1/h}\right)^2} \hat{u}^{\mathrm{T}} \cdot \nabla \tilde{b}_i^{1/h} \cdot \nabla \bar{g}_i \cdot \hat{u}\end{aligned}$$

Using $g_i = c_i \cdot b_i$ we get:

$$\begin{aligned}&\hat{u}^{\mathrm{T}} \cdot \nabla^2 G \cdot \hat{u} = \\ &= g_i \cdot \hat{u}^{\mathrm{T}} \cdot \nabla^2 \bar{g}_i \cdot \hat{u} + v_i \cdot \bar{g}_i \cdot c_i + \tfrac{\bar{g}_i \cdot g_i}{c_i} \cdot \hat{u}^{\mathrm{T}} \cdot A_i \cdot \hat{u} - 2 g_i \tfrac{\lambda}{c_i \left(b_i + \tilde{b}_i^{1/h}\right)^2} \hat{u}^{\mathrm{T}} \cdot \nabla \tilde{b}_i^{1/h} \cdot \nabla \bar{g}_i \cdot \hat{u} = \\ &= g_i \cdot \xi_i + v_i \cdot \bar{g}_i \cdot c_i\end{aligned}$$

where

$$\xi_i = \hat{u}^{\mathrm{T}} \cdot \nabla^2 \bar{g}_i \cdot \hat{u} + \frac{\bar{g}_i}{c_i} \cdot \hat{u}^{\mathrm{T}} \cdot A_i \cdot \hat{u} - 2 \frac{\lambda}{c_i \left(b_i + \tilde{b}_i^{1/h}\right)^2} \hat{u}^{\mathrm{T}} \cdot \nabla \tilde{b}_i^{1/h} \cdot \nabla \bar{g}_i \cdot \hat{u}$$

So eq. (11) becomes:

$$\frac{G^2}{\gamma_d^{k-1}} \hat{u}^{\mathrm{T}} \left(\nabla^2 \hat{\varphi}\right)(q)\, \hat{u} = \left(2kG - v_i \cdot \bar{g}_i \cdot \gamma_d \cdot c_i\right) + g_i \cdot \left(\gamma_d \cdot \eta_i - \gamma_d \cdot \xi_i\right) \tag{16}$$

Taking the inner product of both sides of eq. (7) with $\nabla \gamma_d$, we get:

$$4Gk = \nabla \gamma_d \nabla G = \bar{g}_i \nabla g_i \cdot \nabla \gamma_d + g_i \nabla \bar{g}_i \cdot \nabla \gamma_d \tag{17}$$

Substituting $2Gk$ from eq. (17) in eq. (16) we get:

$$\frac{G^2}{\gamma_d^{k-1}} \hat{u}^{\mathrm{T}} \left(\nabla^2 \hat{\varphi}\right)(q)\, \hat{u} = \bar{g}_i \left(\frac{1}{2} \nabla g_i \cdot \nabla \gamma_d - v_i \cdot c_i \cdot \gamma_d\right) + g_i \cdot \left(\gamma_d \cdot \eta_i - \gamma_d \cdot \xi_i + \nabla \bar{g}_i \cdot \nabla \gamma_d\right)$$

Using eq. (5), we get:

$$\begin{aligned}\frac{G^2}{\gamma_d^{k-1}} \hat{u}^{\mathrm{T}} \left(\nabla^2 \hat{\varphi}\right)(q)\, \hat{u} = \bar{g}_i \cdot c_i \left(\frac{1}{2} \nabla b_i \cdot \nabla \gamma_d - v_i \cdot \gamma_d\right) + \\ g_i \cdot \left(\gamma_d \cdot \eta_i - \gamma_d \cdot \xi_i - \sigma_i + \nabla \bar{g}_i \cdot \nabla \gamma_d\right)\end{aligned} \tag{18}$$

where

$$\sigma_i = \frac{\lambda \bar{g}_i}{2 c_i \left(b + \tilde{b}^{1/h}\right)^2} \left(\nabla b + \nabla \tilde{b}^{1/h}\right) \cdot \nabla \gamma_d$$

Setting

$$\mu_i = \frac{1}{2}\nabla b_{R_i} \cdot \nabla\gamma_d - v_i \cdot \gamma_d$$

then eq. (18) becomes:

$$\frac{G^2}{\gamma_d^{k-1}}\hat{u}^{\mathrm{T}}\left(\nabla^2\hat{\varphi}\right)(q)\,\hat{u} = \bar{g}_i c_i \mu_i + g_{R_i}\left(\gamma_d\eta_i - \gamma_d\xi_i - \sigma_i + \nabla\bar{g}_i\nabla\gamma_d\right) \tag{19}$$

The second term of eq. (19) is proportional to g_{R_i} and can be made arbitrarily small by a suitable choice of ε but can still be positive, so the first term should be strictly negative. We will need the following lemma to proceed with our analysis:

Lemma 1.

$$\max_{q\in F_0}(\mu_i) = (x+a)\cdot(x-a/(m-1))\cdot(m-1)/m$$

where $x = \sqrt{\varepsilon + \sum(r_i+r_j)^2}$ *and* $a = \sqrt{q_d^T P_{R_i} q_d}$

Proof.

$$\mu_i = \nabla b_{R_i} \cdot \nabla\gamma_d/2 - v_i \cdot \gamma_d \le 2f(q)$$

where

$$f(q) = q^T \cdot P_{R_i} \cdot q - q^T \cdot P_{R_i} \cdot q_d - (q-q_d)^T \cdot (q-q_d)$$

Thus

$$\nabla f(q) = 2P_{R_i} \cdot q - P_{R_i} \cdot q_d - 2(q-q_d)$$

If q_c is a critical point, then: $\nabla f(q_c) = 0$. Solving for q_c , we get:

$$q_c = 1/2 \cdot (P_{R_i} - I)^{-1} \cdot (P_{R_i} - 2I) \cdot q^d$$

But for the worst-case scenario (This is when the proximity relation is a complete graph)

$$(P_{R_i} - I)^{-1} = 1/(m-1) \cdot P_{R_i} - I$$

with

$$P_{R_i} = m \cdot I - \begin{bmatrix} 1 & \cdots & 1 \end{bmatrix}^T \begin{bmatrix} 1 & \cdots & 1 \end{bmatrix}$$

So

$$q_c = (I - P_{R_i} \cdot 1/(2m-2))\,q^d$$

and

$$f(q_c) = -m/(4m-4) < 0$$

The Hessian of $f(q)$ is:

$$\nabla^2 f(q) = 2(P_{R_i} - I)$$

It can be verified that $P_{R_i} - I$ has eigenvalues:

1. $\lambda = m-1$ of multiplicity $(m-1)\,D$, where D is the workspace dimension, and

2. $\lambda = -1$ of multiplicity D.

That means that $f(q)$ decreases only along D dimensions about q_c and increases along the $(m-1)\,D$ remaining (for some appropriate coordinate system), which in turn means that q_c is a saddle. We are interested in finding the maximum value that $f(q)$ may attain under the constraint that $b_{R_i} \leq \varepsilon$. We form the constraint function:

$$g(q) = q^T \cdot P_{R_i} \cdot q - \varepsilon - \sum_{\{l,j\}\in R_i} (r_l + r_j)^2 \leq 0$$

Since g is convex ($\nabla^2 g(q) = 2 \cdot P_{R_i} > 0$) and q_c is a saddle point of f , $f(q)$ will attain its maximum and minimum values over the constraint's boundary, $g(q) = 0$. This can be formulated as a nonlinear optimization problem:

$$\max_{q\in U} (f(q))$$

where

$$f(q) = q^T \cdot P_{R_i} \cdot q - q^T \cdot P_{R_i} \cdot q_d - (q - q_d)^T \cdot (q - q_d)$$

and

$$U = \left\{ q : g(q) = q^T \cdot P_{R_i} \cdot q - \varepsilon - \sum_{\{l,j\}\in R_i} (r_l + r_j)^2 \leq 0 \right\}$$

If

$$q^* = \arg\max_{q\in U} (f(q))$$

then, according to Kuhn Tucker conditions, there exists a $\rho \geq 0$ such that:

$$\nabla f(q^*) - \rho \nabla g(q^*) = 0 \tag{20}$$

$$\rho \cdot g(q^*) = 0 \tag{21}$$

$$g(q^*) \leq 0 \tag{22}$$

$$\rho \geq 0 \tag{23}$$

From eq. (20) we have:

$$2P_{R_i} \cdot q^* - P_{R_i} \cdot q_d - 2(q^* - q_d) - 2\rho \cdot P_{R_i} \cdot q^* = 0$$

Solving for q^*, we get

$$q^* = \frac{1}{2} \left(I + (\rho - 1) \cdot P_{R_i}\right)^{-1} (2I - P_{R_i})\, q_d$$

One can easily verify that:

$$\left(I+(\rho-1)\cdot P_{R_i}\right)^{-1}=\left(I-P_{R_i}\cdot(\rho-1)/(1+(\rho-1)\,m)\right)$$

and

$$q^*=\frac{1}{2}\cdot\left(2I-P_{R_i}\cdot(2\rho-1)/(1+(\rho-1)\,m)\right)q_d$$

As discussed above, the constraint should be activated, so $\rho>0$ and from eq. (21) we get:

$$g\left(q^*\right)=0$$

Solving for ρ we get:

$$\rho_{1,2}=\left(2\,(m-1)\pm(m-2)\,a/x\right)/(2m)$$

Both ρ_1 , ρ_2 could be made positive so by substituting in q^* we have:

$$^{+}q^*=\left(I-P_{R_i}(a+x)/(ma)\right)q_d$$

and

$$^{-}q^*=\left(I-P_{R_i}(a-x)/(ma)\right)q_d$$

where $^{+}q^*$, $^{-}q^*$ are the values of q^* for ρ_1, ρ_2 respectively. Examining the terms of $f\,(q)$, we have:

1. $q^T P_{R_i} q=x^2$ for both $^{+}q^*, ^{-}q^*$
2. $q^T P_{R_i} q_d=-ax$ for $^{+}q^*$
3. $q^T P_{R_i} q_d=ax$ for $^{-}q^*$
4. $\left(q-q_d\right)^T\left(q-q_d\right)=\left(a+x\right)^2/m$ for $^{+}q^*$ and
5. $\left(q-q_d\right)^T\left(q-q_d\right)=\left(a-x\right)^2/m$ for $^{-}q^*$

After substituting in $f\,(q)$, we get:

$$f\left(^{+}q^*\right)=x^2+ax-\left(a+x\right)^2/m=\left(x+a\right)\left(x-a/(m-1)\right)(m-1)/m$$

and

$$f\left(^{-}q^*\right)=x^2-ax-\left(a-x\right)^2/m=\left(x+a/(m-1)\right)\left(x-a\right)(m-1)/m$$

Then $f\,(^{+}q^*)<0$ for

$$-a<x<a/(m-1)$$

and $f\left({}^{-}q^{*}\right) < 0$ for

$$-a/(m-1) < x < a$$

We can observe that $f\left({}^{+}q^{*}\right) = f\left({}^{-}q^{*}\right)$ for $x = 0$ and since we are interested for $x > 0$, it holds that $f\left({}^{+}q^{*}\right) > f\left({}^{-}q^{*}\right)$ since

$$f\left({}^{+}q^{*}\right) - f\left({}^{-}q^{*}\right) = 2a\left(m-2\right)x/m > 0, \forall x > 0, m > 2$$

Therefore, by choosing $f\left({}^{+}q^{*}\right)$ we have the result:

$$\max_{q \in F_0}\left(\mu_i\right) = (x+a) \cdot (x - a/(m-1)) \cdot (m-1)/m$$

and the proof of Lemma 1 is complete. □

So according to *Lemma 1*, for μ_i to be negative, it is sufficient to make sure that:

$$\varepsilon < \frac{1}{(m-1)^2} \cdot q_d^T P_{R_H} q_d - \sum_{\{l,j\} \in R_H} \left(r_l + r_j\right)^2 = \ldots$$

$$\ldots = \frac{1}{(m-1)^2} \sum_{\{l,j\} \in R_H} \left\|q_d^l - q_d^j\right\|^2 - \sum_{\{l,j\} \in R_H} \left(r_l + r_j\right)^2 = \varepsilon_0$$

Another constraint arises from the fact that $\varepsilon > 0$. So for a valid workspace it will be:

$$\sum_{\{l,j\} \in R_H} \left\|q_d^l - q_d^j\right\|^2 > (m-1)^2 \cdot \sum_{\{l,j\} \in R_H} \left(r_l + r_j\right)^2$$

where R_H is the highest level relation. □

Proposition 5. *There exists $\varepsilon_1 > 0$ and $h_1 > 0$, such that the critical points of $\hat{\varphi}$ are non-degenerate as long as $\varepsilon < \varepsilon_1$ and $h > h_1$ (Morse Property)*

Proof. Following the line of thought presented in [6], to prove that $\hat{\varphi}$ is non-degenerate, we need to prove that the quadratic form associated to the orthogonal complement of $\mathcal{N}_q = span\left\{\hat{\mathbf{u}}\right\}$ is positive definite. Since $\nabla b_i \perp \hat{\mathbf{u}}$ we need to prove that $\tilde{\mathbf{u}}^T\left(\nabla^2\varphi\right)\tilde{\mathbf{u}} > 0$, where $\tilde{\mathbf{u}} = \widehat{\nabla b_i}$. At a critical point from eq. (7) we get:

$$(k \cdot G)^2 \left\|\nabla \gamma_d\right\|^2 = \gamma_d^2 \left\|\nabla G\right\|^2$$

Hence

$$2kG = \frac{\gamma_d}{2kG}\left\|\nabla G\right\|^2$$

Multiplying eq. (10) from both sides with $\tilde{\mathbf{u}}$, we get:

$$\frac{G^2}{\gamma_d^{k-1}}\tilde{u}^T\left(\nabla^2\hat{\varphi}\right)(q)\,\tilde{u} = 2kG + \left(1-\frac{1}{k}\right)\frac{\gamma_d}{G}\tilde{u}^T\cdot\nabla G\cdot\nabla G^T\cdot\tilde{u} - \gamma_d\cdot\tilde{u}^T\cdot\nabla^2 G\cdot\tilde{u}$$
$$= L + M + N$$

where

$$L = \frac{\gamma_d}{2kG}\left\|\nabla G\right\|^2$$
$$M = \left(1-\frac{1}{k}\right)\frac{\gamma_d}{G}\tilde{u}^T\cdot\nabla G\cdot\nabla G^T\cdot\tilde{u} \tag{24}$$
$$N = -\gamma_d\cdot\tilde{u}^T\cdot\nabla^2 G\cdot\tilde{u}$$

Term L:

$$\left\|\nabla G\right\|^2 = \left\|g_i\nabla\bar{g}_i + \bar{g}_i\nabla g_i\right\|^2 =$$
$$= g_i^2\left\|\nabla\bar{g}_i\right\|^2 + 2G\nabla g_i\cdot\nabla\bar{g}_i + \bar{g}_i^2\left\|\nabla g_i\right\|^2$$

Hence

$$L = \frac{\gamma_d}{2kG}\left(g_i^2\left\|\nabla\bar{g}_i\right\|^2 + 2G\nabla g_i\cdot\nabla\bar{g}_i + \bar{g}_i^2\left\|\nabla g_i\right\|^2\right)$$

and denote $L_a = \frac{\gamma_d}{2kG}\left(2G\nabla g_i\nabla\bar{g}_i\right)$

From term M we have:

$$\nabla G\cdot\nabla G^T = \left(g_i\nabla\bar{g}_i + \bar{g}_i\nabla g_i\right)\left(g_i\nabla\bar{g}_i^T + \bar{g}_i\nabla g_i^T\right) =$$
$$= g_i^2\nabla\bar{g}_i\cdot\nabla\bar{g}_i^T + \bar{g}_i^2\nabla g_i\cdot\nabla g_i^T + 2G\left(\nabla g_i\cdot\nabla\bar{g}_i^T\right)_S$$

Then

$$\tilde{u}^T\nabla G\cdot\nabla G^T\tilde{u} = g_i^2\left(\tilde{u}\cdot\nabla\bar{g}_i\right)^2 + \bar{g}_i^2\left(\tilde{u}\cdot\nabla g_i\right)^2 + 2G\left(\tilde{u}\cdot\nabla g_i\right)\cdot\left(\nabla\bar{g}_i\cdot\tilde{u}\right) \tag{25}$$

M can be written as:

$$M = \left(1-\frac{1}{k}\right)\frac{\gamma_d}{G}\cdot\left(g_i^2\left(\tilde{u}\cdot\nabla\bar{g}_i\right)^2 + \bar{g}_i^2\left(\tilde{u}\cdot\nabla g_i\right)^2\right) +$$
$$+2\frac{\gamma_d}{G}G\left(\tilde{u}\cdot\nabla g_i\right)\cdot\left(\nabla\bar{g}_i\cdot\tilde{u}\right) - 2\frac{1}{k}\frac{\gamma_d}{G}G\left(\tilde{u}\cdot\nabla g_i\right)\cdot\left(\nabla\bar{g}_i\cdot\tilde{u}\right)$$

We denote

$$M_a = 2\frac{\gamma_d}{G}G\left(\tilde{u}\cdot\nabla g_i\right)\cdot\left(\nabla\bar{g}_i\cdot\tilde{u}\right)$$

and

$$M_b = -2\frac{1}{k}\frac{\gamma_d}{G}G\left(\tilde{u}\cdot\nabla g_i\right)\cdot\left(\nabla\bar{g}_i\cdot\tilde{u}\right)$$

Let $M_1 = \tilde{u} \cdot \nabla g_i$. Then

$$M_1 = \widehat{\nabla b_i} \cdot \nabla g_i = \widehat{\nabla b_i} \cdot \left(\nabla b_i \left(1 + \frac{\lambda}{b_i + \tilde{b}_i^{1/h}} \right) - \frac{\lambda b_i}{\left(b_i + \tilde{b}_i^{1/h} \right)^2} \left(\nabla b_i + \nabla \tilde{b}_i^{1/h} \right) \right)$$
$$= \|\nabla b_i\| + \frac{\lambda}{\left(b_i + \tilde{b}_i^{1/h} \right)^2} \widehat{\nabla b_i} \cdot \left(\tilde{b}_i^{1/h} \nabla b_i - b_i \nabla \tilde{b}_i^{1/h} \right)$$

We consider the term: $\widehat{\nabla b_i} \cdot \left(\tilde{b}_i^{1/h} \nabla b_i - b_i \nabla \tilde{b}_i^{1/h} \right)$

Then

$$\widehat{\nabla b_i} \cdot \left(\tilde{b}_i^{1/h} \nabla b_i - b_i \nabla \tilde{b}_i^{1/h} \right) \geq \tilde{b}_i^{1/h} \|\nabla b_i\| - b_i \left\| \nabla \tilde{b}_i^{1/h} \right\|$$

but

$$\|\nabla b_i\| = 2 \sqrt{b_i + \sum_{\{l,j\} \in R_i} (r_l + r_j)^2}$$

Hence

$$\widehat{\nabla b_i} \cdot \left(\tilde{b}_i^{1/h} \nabla b_i - b_i \nabla \tilde{b}_i^{1/h} \right) \geq 2 \tilde{b}_i^{1/h} \sqrt{b_i + \sum_{\{l,j\} \in R_i} (r_l + r_j)^2} - b_i \left\| \nabla \tilde{b}_i^{1/h} \right\|$$
$$\geq 2 \tilde{b}_i^{1/h} \sqrt{\sum_{\{l,j\} \in R_i} (r_l + r_j)^2} - \varepsilon \left\| \nabla \tilde{b}_i^{1/h} \right\| = \tilde{b}_i^{1/h} \left(2 \sqrt{\sum_{\{l,j\} \in R_i} (r_l + r_j)^2} - \varepsilon \left\| \frac{\nabla \tilde{b}_i^{1/h}}{\tilde{b}_i^{1/h}} \right\| \right)$$

but

$$\left\| \frac{\nabla \tilde{b}_i^{1/h}}{\tilde{b}_i^{1/h}} \right\| = \left\| \sum_{\mu \in R_i^C} \frac{\nabla b_\mu^{1/h}}{b_\mu^{1/h}} \right\| < \sum_{\mu \in R_i^C} \frac{1}{h} \frac{b_\mu^{1/h-1}}{b_\mu^{1/h}} \left\| \nabla b_\mu \right\| =$$
$$= \sum_{\mu \in R_i^C} \frac{1}{h \cdot b_\mu} \left\| \nabla b_\mu \right\| < \frac{1}{h \cdot \varepsilon} \sum_{\mu \in R_i^C} \left\| \nabla b_\mu \right\|$$

so

$$\widehat{\nabla b_i} \cdot \left(\tilde{b}_i^{1/h} \nabla b_i - b_i \nabla \tilde{b}_i^{1/h} \right) > \tilde{b}_i^{1/h} \left(2 \sqrt{\sum_{\{l,j\} \in R_i} (r_l + r_j)^2} - \varepsilon \frac{1}{h \cdot \varepsilon} \sum_{\mu \in R_i^C} \left\| \nabla b_\mu \right\| \right)$$
$$= \tilde{b}_i^{1/h} \left(2 \sqrt{\sum_{\{l,j\} \in R_i} (r_l + r_j)^2} - \frac{1}{h} \sum_{\mu \in R_i^C} \left\| \nabla b_\mu \right\| \right)$$

For this to be positive, it must be:

$$h > \frac{1}{2} \cdot \frac{\max \left(\sum_{\mu \in R_i^C} \left\| \nabla b_\mu \right\| \right)}{\min \left(\sqrt{\sum_{\{l,j\} \in R_i} (r_l + r_j)^2} \right)} = h_1 \tag{26}$$

So choosing h according to (26) we have that:

$$M_1 = \widehat{\nabla b_i} \cdot \nabla g_i \geq \|\nabla b_i\|$$

and of course:

$$\|\nabla g_i\| \geq \widehat{\nabla b_i} \cdot \nabla g_i \geq \|\nabla b_i\| \tag{27}$$

Let us examine the term:

$$\begin{aligned}&L_a + M_b = \tfrac{\gamma_d}{2kG}\left(2G\nabla g_i \nabla \bar{g}_i\right) - 2\tfrac{1}{k}\tfrac{\gamma_d}{G} G\left(\tilde{u}\cdot\nabla g_i\right)\cdot\left(\nabla\bar{g}_i\cdot\tilde{u}\right)\\ &= \tfrac{\gamma_d}{k}\left(\nabla g_i \nabla\bar{g}_i - 2\left(\tilde{u}\cdot\nabla g_i\right)\cdot\left(\nabla\bar{g}_i\cdot\tilde{u}\right)\right)\end{aligned}$$

$$\begin{aligned}&\nabla g_i \nabla\bar{g}_i - 2\left(\tilde{u}\cdot\nabla g_i\right)\cdot\left(\nabla\bar{g}_i\cdot\tilde{u}\right) = \nabla\bar{g}_i\left(\nabla g_i - 2\left(\tilde{u}\cdot\nabla g_i\right)\cdot\tilde{u}\right) =\\ &-\nabla\bar{g}_i\left(2\left(\tilde{u}\cdot\nabla g_i\right)\cdot\tilde{u} - \nabla g_i\right) \geq -\|\nabla\bar{g}_i\|\left\|\left(2\left(\tilde{u}\cdot\nabla g_i\right)\cdot\tilde{u} - \nabla g_i\right)\right\|\end{aligned}$$

but

$$\left\|\left(2\left(\tilde{u}\cdot\nabla g_i\right)\cdot\tilde{u} - \nabla g_i\right)\right\|^2 = 4\left\|\tilde{u}\cdot\nabla g_i\right\|^2 - 4\left(\tilde{u}\cdot\nabla g_i\right)\cdot\left(\tilde{u}\cdot\nabla g_i\right) + \|\nabla g_i\|^2$$

So

$$\left\|\left(2\left(\tilde{u}\cdot\nabla g_i\right)\cdot\tilde{u} - \nabla g_i\right)\right\| = \|\nabla g_i\|$$

Hence

$$\nabla g_i \nabla\bar{g}_i - 2\left(\tilde{u}\cdot\nabla g_i\right)\cdot\left(\nabla\bar{g}_i\cdot\tilde{u}\right) \geq -\|\nabla\bar{g}_i\|\,\|\nabla g_i\|$$

So

$$L_a + M_b \geq -\frac{\gamma_d}{k}\|\nabla\bar{g}_i\|\,\|\nabla g_i\|$$

We can then write the term $L + M_b$ as:

$$\begin{aligned}&L + M_b \geq \tfrac{\gamma_d}{2kG}\left(g_i^2\|\nabla\bar{g}_i\|^2 + \bar{g}_i^2\|\nabla g_i\|^2 - 2G\|\nabla\bar{g}_i\|\,\|\nabla g_i\|\right)\\ &= \tfrac{\gamma_d}{2kG}\left(g_i\|\nabla\bar{g}_i\| - \bar{g}_i\|\nabla g_i\|\right)^2 \geq 0\end{aligned}$$

which is non-negative and can be neglected.

Term N:

$$N = -\gamma_d\cdot\tilde{u}^T\cdot\nabla^2 G\cdot\tilde{u}$$

Expanding:

$$\tilde{u}^T\cdot\nabla^2 G\cdot\tilde{u} = \tilde{u}^T\cdot\left(g_i\nabla^2\bar{g}_i + \bar{g}_i\nabla^2 g_i\right)\cdot\tilde{u} + 2\left(\tilde{u}^T\cdot\nabla\bar{g}_i\right)\cdot\left(\tilde{u}^T\cdot\nabla g_i\right)$$

Notice that the second term is cancelled with M_a. Using equation (27), we can write (since $k > 1$):

$$\frac{G^2}{\gamma_d^{k-1}}\tilde{u}^T\left(\nabla^2\hat{\varphi}\right)(q)\,\tilde{u} \geq \frac{\gamma_d}{g_i}\left(\left(1-\frac{1}{k}\right)\bar{g}_i\|\nabla b_i\|^2 - g_i^2\tilde{u}^T\nabla^2\bar{g}_i\tilde{u} - g_i\bar{g}_i\tilde{u}^T\nabla^2 g_i\tilde{u}\right)$$

We will now proceed by examining the term:

$$\tilde{u}^T \cdot \nabla^2 g_i \cdot \tilde{u}$$

From eq. (6) we get:

$$\tilde{u}^T \cdot \nabla^2 g_i \cdot \tilde{u} = c_i \cdot \tilde{u}^T \cdot \nabla^2 b_i \cdot \tilde{u} - s_i \cdot \tilde{u}^T \cdot \nabla b_i \cdot \left(\nabla b_i + \nabla \tilde{b}_i^{1/h}\right)^T \cdot \tilde{u} + b_i \cdot \tilde{u}^T \cdot A_i \cdot \tilde{u}$$

where

$$s_i = \frac{2\lambda}{\left(b_i + \tilde{b}_i^{1/h}\right)^2}$$

from

$$\begin{aligned}
&\tilde{u}^T \cdot \nabla b_i \cdot \left(\nabla b_i + \nabla \tilde{b}_i^{1/h}\right)^T \cdot \tilde{u} = \|\nabla b_i\|^2 + \|\nabla b_i\| \left(\nabla \tilde{b}_i^{1/h}\right) \cdot \widehat{\nabla b_i} = \\
&\|\nabla b_i\|^2 + \nabla b_i \cdot \nabla \tilde{b}_i^{1/h} \geq \|\nabla b_i\|^2 - \|\nabla b_i\| \cdot \left\|\nabla \tilde{b}_i^{1/h}\right\|
\end{aligned}$$

For the last term we have:

$$\begin{aligned}
&\|\nabla b_i\| - \left\|\nabla \tilde{b}_i^{1/h}\right\| = 2\sqrt{b_i + \sum_{\{l,j\} \in R_i} (r_l + r_j)^2} - \tilde{b}_i^{1/h} \left\| \frac{\nabla \tilde{b}_i^{1/h}}{\tilde{b}_i^{1/h}} \right\| \\
&> 2\sqrt{b_i + \sum_{\{l,j\} \in R_i} (r_l + r_j)^2} - \frac{\tilde{b}_i^{1/h}}{h \cdot \varepsilon} \sum_{\mu \in R_i^C} \|\nabla b_\mu\| \geq 0
\end{aligned} \tag{28}$$

This is guaranteed to be positive, as long as:

$$h \geq \frac{\max\left\{1,\ \tilde{b}_i\right\} \cdot \max\left\{\sum_{\mu \in R_i^C} \|\nabla b_\mu\|\right\}}{2 \cdot \varepsilon \cdot \min\left\{\sqrt{\sum_{\{l,j\} \in R_i} (r_l + r_j)^2}\right\}} = h_2(\varepsilon)$$

Hence the term $s_i \cdot \tilde{u}^T \cdot \nabla b_i \cdot \left(\nabla b_i + \nabla \tilde{b}_i^{1/h}\right)^T \cdot \tilde{u} > 0$ and can be neglected.

Recalling that:

$$A_i = \lambda \left(2 \frac{\left(\nabla b_i + \nabla \tilde{b}_i^{1/h}\right)\left(\nabla b_i + \nabla \tilde{b}_i^{1/h}\right)^T}{\left(b_i + \tilde{b}_i^{1/h}\right)^3} - \frac{\left(\nabla^2 b_i + \nabla^2 \tilde{b}_i^{1/h}\right)}{\left(b_i + \tilde{b}_i^{1/h}\right)^2} \right)$$

for the term $b_i \cdot \tilde{u}^T \cdot A_i \cdot \tilde{u}$ we have:

$$\widehat{\nabla b_i}^T \left(\nabla b_i + \nabla \tilde{b}_i^{1/h}\right)\left(\nabla b_i + \nabla \tilde{b}_i^{1/h}\right)^T \widehat{\nabla b_i} = \left(\|\nabla b_i\| + \widehat{\nabla b_i} \cdot \nabla \tilde{b}_i^{1/h}\right)^2 < 4\|\nabla b_i\|^2$$

because of (28). For $\left(b_i+\tilde{b}_i^{1/h}\right)^3$, with $\varepsilon<1$ we have:

$$\left(b_i+\tilde{b}_i^{1/h}\right)^3>\tilde{b}_i^{3/h}>\varepsilon^{3n_R/h}$$

where n_R+1 is the number of relations in the level with maximum relations. With

$$h>3n_R=h_3$$

we have:

$$\left(b_i+\tilde{b}_i^{1/h}\right)^3>\varepsilon$$

Hence:

$$\tilde{u}^T\cdot A_i\cdot\tilde{u}<\frac{8\lambda}{\varepsilon}\left\|\nabla b_i\right\|^2-\frac{s_i}{2}\tilde{u}^T\nabla^2 b_i\tilde{u}-\frac{s_i}{2}\tilde{u}^T\nabla^2\tilde{b}_i^{1/h}\tilde{u}$$

and

$$\tilde{u}^T\cdot\nabla^2 g_i\cdot\tilde{u}<c_i\cdot\tilde{u}^T\cdot\nabla^2 b_i\cdot\tilde{u}+8\lambda\left\|\nabla b_i\right\|^2-b_i\frac{s_i}{2}\tilde{u}^T\nabla^2 b_i\tilde{u}-b_i\frac{s_i}{2}\tilde{u}^T\nabla^2\tilde{b}_i^{1/h}\tilde{u}$$

Hence

$$\begin{aligned}&\tfrac{G^2}{\gamma_d^{k-1}}\tilde{u}^T\left(\nabla^2\hat{\varphi}\right)(q)\,\tilde{u}\geq\tfrac{\gamma_d}{g_i}\left(\left(1-\tfrac{1}{k}\right)\bar{g}_i\left\|\nabla b_i\right\|^2-g_i^2\tilde{u}^T\nabla^2\bar{g}_i\tilde{u}-\right.\\&\left.-g_i\bar{g}_i\left(c_i\cdot\tilde{u}^T\cdot\nabla^2 b_i\cdot\tilde{u}+8\lambda\left\|\nabla b_i\right\|^2-b_i\tfrac{s_i}{2}\tilde{u}^T\nabla^2 b_i\tilde{u}-b_i\tfrac{s_i}{2}\tilde{u}^T\nabla^2\tilde{b}_i^{1/h}\tilde{u}\right)\right)\end{aligned}$$

From the term $\tilde{u}^T\cdot\nabla^2 b_i\cdot\tilde{u}$, following the analysis presented in Proposition 4, we get:

$$\tilde{u}^T\nabla^2 b_i\tilde{u}=2+\frac{q^T\cdot P\cdot R\cdot q}{q^T\cdot P\cdot P\cdot q}$$

Eq. (13) in this case becomes:

$$\tilde{u}^T\nabla^2 b_i\tilde{u}=2+\frac{\sum\limits_{i=1}^{T}\sum\limits_{j=1}^{n_{tree_t}}\left(\left(c_{j,i}^2-1\right)\sum\limits_{m=1}^{c_{j,i}}\left(q^\perp\right)^T\cdot D_{p_j,q_m}\cdot q^\perp-c_{j,i}\sum\limits_{m=1}^{c_{j,i}-1}\sum\limits_{n=m+1}^{c_{j,i}}\left(q^\perp\right)^T\cdot D_{p_m,q_n}\cdot q^\perp\right)}{\sum\limits_{i=1}^{T}\sum\limits_{j=1}^{n_{tree_t}}\left(\left(c_{j,i}+1\right)\sum\limits_{m=1}^{c_{j,i}}\left(q^\perp\right)^T\cdot D_{p_j,q_m}\cdot q^\perp-\sum\limits_{m=1}^{c_{j,i}-1}\sum\limits_{n=m+1}^{c_{j,i}}\left(q^\perp\right)^T\cdot D_{p_m,q_n}\cdot q^\perp\right)}$$

Using eqns. (14, 15) and the fact that $1\leq c_{j,i}\leq m-1$, we get:

$$2<\tilde{u}^T\nabla^2 b_i\tilde{u}<2+\frac{(m-2)\left(m^2-1\right)}{2m}\cdot\frac{r_{\max}}{r_{\min}}$$

Hence:

$$\begin{aligned}&\tfrac{G^2}{\gamma_d^{k-1}}\tilde{u}^T\left(\nabla^2\hat{\varphi}\right)(q)\,\tilde{u}\geq\tfrac{\gamma_d}{g_i}\left(\left(1-\tfrac{1}{k}\right)\bar{g}_i\left\|\nabla b_i\right\|^2-g_i^2\tilde{u}^T\nabla^2\bar{g}_i\tilde{u}-\right.\\&\left.-g_i\bar{g}_i\left(c_i\left(2+\tfrac{(m-2)\left(m^2-1\right)}{2m}\cdot\tfrac{r_{\max}}{r_{\min}}\right)+8\lambda\left\|\nabla b_i\right\|^2-b_is_i-b_i\tfrac{s_i}{2}\tilde{u}^T\nabla^2\tilde{b}_i^{1/h}\tilde{u}\right)\right)\end{aligned}$$

$$\geq \tfrac{\gamma_d}{g_i}\left(\left(1-\tfrac{1}{k}\right)\bar{g}_i\left\|\nabla b_i\right\|^2 - g_i^2\tilde{u}^T\nabla^2\bar{g}_i\tilde{u} - \right.$$
$$\left. -g_i\bar{g}_i\left(c_i\left(2+\tfrac{(m-2)\left(m^2-1\right)}{2m}\cdot\tfrac{r_{\max}}{r_{\min}}\right)+8\lambda\left\|\nabla b_i\right\|^2 - b_i\tfrac{s_i}{2}\tilde{u}^T\nabla^2\tilde{b}_i^{1/h}\tilde{u}\right)\right)$$

For c_i we have:

$$c_i = 1+\frac{\lambda}{b_i+\tilde{b}_i^{1/h}} \leq 1+\frac{\lambda}{\tilde{b}_i^{1/h}} < 1+\frac{\lambda}{\varepsilon^{n_R/h}} \leq 1+\frac{\lambda}{\varepsilon^{1/3}}$$

Hence:

$$\tfrac{G^2}{\gamma_d^{k-1}}\tilde{u}^T\left(\nabla^2\hat{\varphi}\right)(q)\,\tilde{u} \geq \tfrac{\gamma_d}{g_i}\left(\left(1-\tfrac{1}{k}\right)\bar{g}_i\left\|\nabla b_i\right\|^2 - g_i^2\tilde{u}^T\nabla^2\bar{g}_i\tilde{u} - \right.$$
$$\left. -g_i\bar{g}_i\left(\left(1+\tfrac{\lambda}{\varepsilon^{1/3}}\right)\left(2+\tfrac{(m-2)\left(m^2-1\right)}{2m}\cdot\tfrac{r_{\max}}{r_{\min}}\right)+8\lambda\left\|\nabla b_i\right\|^2 - b_i\tfrac{s_i}{2}\tilde{u}^T\nabla^2\tilde{b}_i^{1/h}\tilde{u}\right)\right)$$

Rearranging terms:

$$\tfrac{G^2}{\gamma_d^{k-1}}\tilde{u}^T\left(\nabla^2\hat{\varphi}\right)(q)\,\tilde{u} \geq$$
$$\tfrac{\gamma_d}{g_i}\left(\bar{g}_i\sum_{j=1}^{3}\left(\tfrac{1}{6}\left(1-\tfrac{1}{k}\right)\left\|\nabla b_i\right\|^2-\varepsilon\cdot K_j\right)+\left(\tfrac{1}{2}\left(1-\tfrac{1}{k}\right)\left\|\nabla b_i\right\|^2-\varepsilon^2\tilde{u}^T\nabla^2\bar{g}_i\tilde{u}\right)\right) \tag{29}$$

where

$$K_1 = \frac{1+\lambda}{\varepsilon^{\frac{1}{3}}}\left(2+\frac{(m-2)\left(m^2-1\right)}{2m}\cdot\frac{r_{\max}}{r_{\min}}\right)$$

$$K_2 = 8\lambda\left\|\nabla b_i\right\|^2$$

$$K_3 = \lambda\varepsilon^{\frac{1}{3}}\left|\tilde{u}^T\nabla^2\tilde{b}_i^{1/h}\tilde{u}\right|$$

Assuming that $k>2$, and noting that $\min\left(\left\|\nabla b_i\right\|^2\right) = 4\sum\limits_{\{l,j\}\in R_i}\left(r_l+r_j\right)^2$, then for both the right hand side terms of ineq. (29) to be positive, the sufficient conditions are:

$$\varepsilon < \left(\frac{2\sum\limits_{\{l,j\}\in R_i}\left(r_l+r_j\right)^2}{3\left(1+\lambda\right)\left(2+\frac{(m-2)\left(m^2-1\right)}{2m}\cdot\frac{r_{\max}}{r_{\min}}\right)}\right)^{\frac{3}{2}} = \varepsilon_2$$

$$\varepsilon < \frac{1}{48\lambda} = \varepsilon_3$$

$$\varepsilon < \left(\frac{2 \sum\limits_{\{l,j\} \in R_i} (r_l + r_j)^2}{3\lambda \cdot \max\left(\left| \tilde{\mathbf{u}}^T \nabla^2 \tilde{b}^{1/h} \tilde{\mathbf{u}} \right| \right)} \right)^{\frac{3}{2}} = \varepsilon_4$$

So to render (29) positive it is sufficient to choose an $\varepsilon < \varepsilon_1 = \min\{1, \varepsilon_2, \varepsilon_3, \varepsilon_4\}$ and an $h > h_0 = \max\{h_1, h_2(\varepsilon), h_3\}$.

□

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APPENDIX A

Proposition A-1. *The product*

$$P_{R_k}^T \cdot P_{R_k}$$

with $P_{R_k} = \sum_{\{i,j\} \in R_k} D_{i,j}$ *is a linear combination of* D_{ij}*'s.*

Proof. We denote by $\mathbf{S}$ the set of all possible couples that can be achieved by the team we are considering. Let

$$S = \sum_{\{i,j\} \in \mathbf{S}} D_{i,j}$$

denote the matrix representation of $\mathbf{S}$. Then S has the form:

$$S = \begin{pmatrix} (m-1) & -1 & \cdots & -1 \\ -1 & \ddots & -1 & \vdots \\ \vdots & -1 & \ddots & -1 \\ -1 & \cdots & -1 & (m-1) \end{pmatrix} = -\begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \begin{bmatrix} 1 & \cdots & 1 \end{bmatrix} + m \cdot I$$

The matrix S has the following properties:

$$S \cdot S^T = S \cdot S = \left(-\begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \begin{bmatrix} 1 & \cdots & 1 \end{bmatrix} + m \cdot I \right) \cdot \left(-\begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \begin{bmatrix} 1 & \cdots & 1 \end{bmatrix} + m \cdot I \right)$$

$$= m \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \begin{bmatrix} 1 & \cdots & 1 \end{bmatrix} - 2m \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \begin{bmatrix} 1 & \cdots & 1 \end{bmatrix} + m^2 \cdot I = m \cdot S$$

If

$$H = \sum_{\{i,j\} \in U} D_{i,j}$$

with $U \subset \mathbf{S}$, then

$$H \cdot S = \left(D_{\{i,j\}_1} + \ldots + D_{\{i,j\}_v} \right) \cdot \left(-\begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \begin{bmatrix} 1 & \cdots & 1 \end{bmatrix} + m \cdot I \right) = m \cdot H$$

since $D_{i,j} \cdot \begin{bmatrix} 1 & \ldots & 1 \end{bmatrix}^T = 0$ for every couple $\{i,j\}$.

The set **S** can be partitioned into two subsets, **P** and **C** such that $S = P + C$. The matrix S can then be written as:

$$S = \sum_{\{i,j\}\in\mathbf{S}} D_{i,j} = P + C$$

where $P = \sum_{\{i,j\}\in P} D_{i,j}$ and $C = \sum_{\{i,j\}\in C} D_{i,j}$. If we denote as

$$p_i = \begin{bmatrix} 0 & \dots & 0 & v & 0 & \dots & 0 \end{bmatrix}^T$$

where $v = 1$ is the i'th element of the array. Then

$$D_{i,j} = (p_i - p_j)(p_i - p_j)^T$$

Obviously

$$p_i^T p_j = 0$$

for

$$i \neq j$$

It will be useful to examine here the product:

$$\begin{aligned}
&(D_{ij} + D_{kl}) \cdot (D_{ij} + D_{kl}) = D_{ij} \cdot D_{ij} + D_{kl} \cdot D_{ij} + D_{ij} \cdot D_{kl} + D_{kl} \cdot D_{kl} = \\
&= (p_i - p_j)(p_i - p_j)^T (p_i - p_j)(p_i - p_j)^T + (p_k - p_l)(p_k - p_l)^T (p_k - p_l)(p_k - p_l)^T + \\
&+ (p_k - p_l)(p_k - p_l)^T (p_i - p_j)(p_i - p_j)^T + (p_i - p_j)(p_i - p_j)^T (p_k - p_l)(p_k - p_l)^T = \\
&= 2\left((p_i - p_j)(p_i - p_j)^T + (p_k - p_l)(p_k - p_l)^T\right) + \\
&+\delta_{ki}\left((p_k - p_l)(p_k - p_j)^T + (p_k - p_j)(p_k - p_l)^T\right) + \\
&+\delta_{lj}\left((p_k - p_l)(p_i - p_l)^T + (p_i - p_l)(p_k - p_l)^T\right) + \\
&-\delta_{kj}\left((p_k - p_l)(p_i - p_k)^T + (p_i - p_k)(p_k - p_l)^T\right) \\
&-\delta_{li}\left((p_k - p_l)(p_l - p_j)^T + (p_l - p_j)(p_k - p_l)^T\right) = \\
&= 2\left((p_i - p_j)(p_i - p_j)^T + (p_k - p_l)(p_k - p_l)^T\right) + \\
&+ (\delta_{ki} + \delta_{lj} + \delta_{kj} + \delta_{li})\left(2p_x p_x^T - \left(p_x p_y^T + p_y p_x^T\right) - \left(p_x p_z^T + p_z p_x^T\right) + \left(p_z p_y^T + p_y p_z^T\right)\right) \\
&= 2D_{ij} + 2D_{kl} + (\delta_{ki} + \delta_{lj} + \delta_{kj} + \delta_{li})(D_{xy} + D_{xz} - D_{yz})
\end{aligned}$$

where:

$$\begin{aligned}
x &= k(\delta_{ki} + \delta_{kj}) + l(\delta_{lj} + \delta_{li}) \\
y &= l(\delta_{ki} + \delta_{kj}) + k(\delta_{lj} + \delta_{li}) \\
z &= j(\delta_{ki} + \delta_{li}) + i(\delta_{lj} + \delta_{kj})
\end{aligned}$$

and δ_{ij} is the Kronecker delta:

$$\delta_{ij} = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}$$

We set

$$L_{k,l,i,j} = 2D_{ij} + 2D_{kl} + (\delta_{ki} + \delta_{lj} + \delta_{kj} + \delta_{li})\,(D_{xy} + D_{xz} - D_{yz})$$

Lets assume we have the matrix P, as above. Then:

$$\begin{aligned}
&P \cdot P^T = P \cdot P = \\
&= \left(D_{1,2} + D_{1,3} \ldots + D_{p_{k-1},q_{k-1}} + D_{p_k,q_k}\right) \cdot \left(D_{1,2} + D_{1,3} \ldots + D_{p_{k-1},q_{k-1}} + D_{p_k,q_k}\right) = \\
&= (D_{12} + D_{13}) \cdot (D_{12} + D_{13}) + \ldots + (D_{1,3} + D_{p_k,q_k}) \cdot (D_{1,3} + D_{p_k,q_k}) + \ldots \\
&\ldots + \left(D_{p_{k-1},q_{k-1}} + D_{p_k,q_k}\right) \cdot \left(D_{p_{k-1},q_{k-1}} + D_{p_k,q_k}\right) - (k-2)\left(\sum_{i=1}^{k} D_{p_i,q_i} \cdot D_{p_i,q_i}\right)
\end{aligned}$$

with k being the number of $D_{i,j}$'s from which P is constructed. Using the results from above we have that:

$$\begin{aligned}
P \cdot P^T &= \sum L_{k,l,i,j} - 2\,(k-2)\left(\sum_{i=1}^{k} D_{p_i,q_i}\right) \\
&= \sum_{i=1}^{k-1} \sum_{j=i+1}^{k} L_{p_i,q_i,p_j,q_j} - 2\,(k-2)\,P \\
&= 2P + \sum_{i=1}^{k-1} \sum_{j=i+1}^{k} \Delta_{p_i,q_i,p_j,q_j}
\end{aligned}$$

where

$$\Delta_{k,l,i,j} = (\delta_{ki} + \delta_{lj} + \delta_{kj} + \delta_{li})\,(D_{xy} + D_{xz} - D_{yz})$$

which is a linear combination of $D_{i,j}$'s. The proof is completed by noting that for the set of relations contained in the product of the P matrices, denoted by $\mathbf{P^2}$, it holds that

$$\mathbf{P} \subseteq \mathbf{P^2} \subseteq \mathbf{S}$$

$\square$

Proposition A-2. *Assume we have*

$$D_{ij} + D_{kl}$$

and

$$(\delta_{ki} + \delta_{lj} + \delta_{kj} + \delta_{li}) = 1$$

We denote by:

$$\begin{aligned}
x &= k\,(\delta_{ki} + \delta_{kj}) + l\,(\delta_{lj} + \delta_{li}) \\
y &= l\,(\delta_{ki} + \delta_{kj}) + k\,(\delta_{lj} + \delta_{li}) \\
z &= j\,(\delta_{ki} + \delta_{li}) + i\,(\delta_{lj} + \delta_{kj})
\end{aligned}$$

Then the relation:

$$\left(D_{xy}+D_{xz}\right)D_{yz}=D_{yz}$$

holds.

Proof.

$$\begin{aligned}
&\left(D_{xy}+D_{xz}\right)D_{yz}=D_{xy}D_{yz}+D_{xz}D_{yz}=\\
&=\left(p_x-p_y\right)\left(p_x-p_y\right)^T\left(p_y-p_z\right)\left(p_y-p_z\right)^T+\left(p_x-p_z\right)\left(p_x-p_z\right)^T\left(p_y-p_z\right)\left(p_y-p_z\right)^T=\\
&=-\left(p_x-p_y\right)\left(p_y-p_z\right)^T+\left(p_x-p_z\right)\left(p_y-p_z\right)^T=\left(p_y-p_z\right)\left(p_y-p_z\right)^T=\\
&=D_{yz}
\end{aligned}$$

□

Proposition A-3. *If P is a relationship matrix and*

$$R=\sum_{i=1}^{k-1}\sum_{j=i+1}^{k}\Delta_{p_i,q_i,p_j,q_j}$$

with Δ_{p_i,q_i,p_j,q_j} defined in proposition (A-1), then the following holds:

$$P\cdot R=P_{tree_1}\cdot R_{tree_1}+\ldots+P_{tree_T}\cdot R_{tree_T}=\sum_{t=1}^{T}\sum_{j=1}^{n_{tree_i}}\left(4\sum_{m=1}^{c_{j,i}}D^t_{p_j,q_m}-3\sum_{m=1}^{c_{j,i}-1}\sum_{n=m+1}^{c_{j,i}}D^t_{p_m,q_n}\right)$$

The indices are defined in the proof.

Proof. For the following analysis it will be helpful to consider the proximity relations as branches of a tree structure. For example the following relationship proximity function:

$$\begin{aligned}
&b=\beta_{(\mathrm{A},\mathrm{B})}+\beta_{(\mathrm{A},C)}+\beta_{(\mathrm{A},D)}+\beta_{(D,E)}+\beta_{(F,G)}+\beta_{(G,H)}=\\
&=q^T\left(D_{AB}+D_{AC}+D_{AD}+D_{DE}+D_{FG}+D_{GH}\right)q-\sum_{\{i,j\}\in\mathbf{D}}\left(r_i+r_j\right)^2
\end{aligned}$$

with

$$\mathbf{D}=\left\{\{A,B\},\{A,C\},\{A,D\},\{D,E\},\{F,G\},\{G,H\}\right\}$$

can be represented by the tree structure in (figure 3). Notice that this relation is represented by two separate trees. The first tree has 4 branches (AB, AC, AD, DE), 2 nodes (A, D) and 1 node-connecting branch (AD) and the second has 2 branches (FG, GH) and 1 node (G). Node A has 3 attached branches node D has 2 and node G has 2. Now let us assume that we have a relationship proximity function that consists of k robot proximity functions. Moreover assume that the relation has T trees that represent it, and each tree has b_T

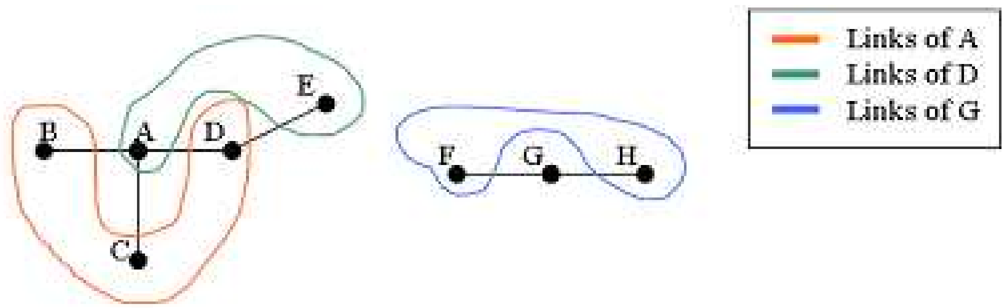

***Figure 3** : Representation of a robot proximity situation. A ... H are robots*

branches, n_T nodes, ncb_T node-connecting branches and each node has $c_{n,T}$ node-attached branches. The respective relationship matrix can be written as:

$$P = P_{tree_1} + \ldots + P_{tree_T}$$

where

$$P_{tree_i} = \left(D_{tree_i \div branch_1} + D_{tree_i \div branch_2} + \ldots + D_{tree_i \div branch_{b_{treei}}} \right)$$

Obviously

$$P_{tree_i} \cdot P_{tree_j} = 0$$

with $i \neq j$. Another useful, more descriptive representation of P_{tree_i} is the following:

$$P_{tree_i} = \left(NB_1^i + NB_2^i + \ldots + NB_{n_{tree_i}}^i \right) + \sum_{k=1}^{ncb_{tree_i}} NC_k^i$$

where NB_j^i is the sum of the branches of tree i that are connected to node j excluding node connecting branches. $NC_k^i \equiv NC_{\{m,n\}}^i$ is the k'th node connecting branch of tree i, which connects the nodes m and n. Here we assume that every node has one node connecting branch with other nodes. If more node connecting branches belong to some node, then NC_k^i can be replaced by the sum of those branches and the analysis won't change. Now let us consider the sum:

$$R = \sum_{i=1}^{k-1} \sum_{j=i+1}^{k} \Delta_{p_i,q_i,p_j,q_j}$$

with

$$\Delta_{k,l,i,j} = (\delta_{ki} + \delta_{lj} + \delta_{kj} + \delta_{li}) \left(D_{xy} + D_{xz} - D_{yz} \right)$$

The above sum can be written again as a sum of 's of the respective couple combinations of the node-attached branches of each node of each tree:

$$R = R_{tree_1} + \ldots + R_{tree_T}$$

where

$$R_{tree_i} = Comb\left(NB_1^i + NC_{\{1,k\}}^i\right) + \ldots + Comb\left(NB_k^i + NC_{\{1,k\}}^i\right) + \ldots$$
$$+ Comb\left(NB_{n_{tree_i}}^i + NC_{\left\{n_{tree_i},v\right\}}^i\right)$$

with

$$Comb\left(D_{p_1q_1} + D_{p_2q_2} + \ldots + D_{p_Zq_Z}\right) = \sum_{i=1}^{Z-1}\sum_{j=i+1}^{Z}\left(D_{xy} + D_{xz} - D_{yz}\right)$$

where

$$\begin{array}{l} x = p_j\left(\delta_{p_jp_i} + \delta_{p_jq_i}\right) + q_j\left(\delta_{q_jq_i} + \delta_{q_jp_i}\right) \\ y = q_j\left(\delta_{p_jp_i} + \delta_{p_jq_i}\right) + p_j\left(\delta_{q_jq_i} + \delta_{q_jp_i}\right) \\ z = q_i\left(\delta_{p_jp_i} + \delta_{q_jp_i}\right) + p_i\left(\delta_{q_jq_i} + \delta_{p_jq_i}\right) \end{array}$$

Note that the arguments of each of the Comb operators in R_{tree_i} are branches of the same node. We are now ready to examine the product:

$$\begin{array}{l} P \cdot R = \left(P_{tree_1} + \ldots + P_{tree_T}\right) \cdot \left(R_{tree_1} + \ldots + R_{tree_T}\right) = \\ = P_{tree_1} \cdot R_{tree_1} + \ldots + P_{tree_T} \cdot R_{tree_T} \end{array}$$

Since any product of the form:

$$P_{tree_i} \cdot R_{tree_j} = 0$$

for $i \neq j$. The product:

$$P_{tree_i} \cdot R_{tree_i} = \left(\left(NB_1^i + NB_2^i + \ldots + NB_{n_{tree_i}}^i\right) + \sum_{k=1}^{ncb_{tree_i}} NC_k^i\right) \cdot$$
$$\cdot\left(Comb\left(NB_1^i + NC_{\{1,k\}}^i\right) + \ldots + Comb\left(NB_k^i + NC_{\{1,k\}}^i\right) + \ldots\right.$$
$$\left.\ldots + Comb\left(NB_{n_{tree_i}}^i + NC_{\left\{n_{tree_i},v\right\}}^i\right)\right)$$
$$= \left(NB_1^i + NC_{\{1,k\}}^i\right) \cdot Comb\left(NB_1^i + NC_{\{1,k\}}^i\right) + \ldots$$

$$\ldots + \left(NB_{n_{tree_i}}^i + NC_{\left\{n_{tree_i},v\right\}}^i\right) \cdot Comb\left(NB_{n_{tree_i}}^i + NC_{\left\{n_{tree_i},v\right\}}^i\right)$$

$$= \sum_{j=1}^{n_{tree_i}} \left(NB_j^i + NC_{\{j,k(j)\}}^i\right) \cdot Comb\left(NB_j^i + NC_{\{j,k(j)\}}^i\right)$$

where v is a node (or a set of nodes) connected with n_{tree_i} and $k(j)$ is a function returning the nodes that are connected with node j. The above holds because

$$NB_j^i \cdot Comb\left(NB_m^i + NC_{\{m,k\}}^i\right) = 0$$

with $j \neq m$, $j \neq k$ and

$$NC_{\{l,n\}}^i \cdot Comb\left(NB_m^i + NC_{\{m,k\}}^i\right) = 0$$

with

$$l \neq m,\ l \neq k,\ n \neq m,\ n \neq k$$

Now we will examine each product in the sum:

$$\begin{aligned}
&\left(NB_j^i + NC_{\{j,k\}}^i\right) \cdot Comb\left(NB_j^i + NC_{\{j,k\}}^i\right) = \\
&= \left(\sum_{m=1}^{c_{j,i}} D_{p_j,q_m}\right) \cdot \left(\sum_{m=1}^{c_{j,i}-1} \sum_{n=m+1}^{c_{j,i}} \left(D_{p_j,q_m} + D_{p_j,q_n} - D_{p_m,q_n}\right)\right) \\
&= \left(\sum_{m=1}^{c_{j,i}} D_{p_j,q_m}\right) \cdot \left((c_{j,i}-1) \sum_{m=1}^{c_{j,i}} D_{p_j,q_m} - \sum_{m=1}^{c_{j,i}-1} \sum_{n=m+1}^{c_{j,i}} D_{p_m,q_n}\right) = \\
&= (c_{j,i}-1) \left(\sum_{m=1}^{c_{j,i}} D_{p_j,q_m}\right)^2 - \left(\sum_{m=1}^{c_{j,i}} D_{p_j,q_m}\right) \left(\sum_{m=1}^{c_{j,i}-1} \sum_{n=m+1}^{c_{j,i}} D_{p_m,q_n}\right)
\end{aligned}$$

The first term, calculated previously gives:

$$\left(\sum_{m=1}^{c_{j,i}} D_{p_j,q_m}\right)^2 = (c_{j,i}+1) \sum_{m=1}^{c_{j,i}} D_{p_j,q_m} - \sum_{m=1}^{c_{j,i}-1} \sum_{n=m+1}^{c_{j,i}} D_{p_m,q_n}$$

since all are connected on the same node. The second term gives:

$$\begin{aligned}
&\left(\sum_{m=1}^{c_{j,i}} D_{p_j,q_m}\right) \left(\sum_{m=1}^{c_{j,i}-1} \sum_{n=m+1}^{c_{j,i}} D_{p_m,q_n}\right) = \\
&= \sum_{m=1}^{c_{j,i}-1} \sum_{n=m+1}^{c_{j,i}} \left(\left(D_{p_j,q_m} + D_{p_j,q_n}\right) \cdot D_{p_m,q_n}\right) = \sum_{m=1}^{c_{j,i}-1} \sum_{n=m+1}^{c_{j,i}} D_{p_m,q_n}
\end{aligned}$$

by use of proposition A.2. Hence:

$$\left(NB_j^i + NC_{\{j,k\}}^i\right) \cdot Comb\left(NB_j^i + NC_{\{j,k\}}^i\right) =$$
$$= (c_{j,i}-1)\left((c_{j,i}+1)\sum_{m=1}^{c_{j,i}} D_{p_j,q_m} - \sum_{m=1}^{c_{j,i}-1}\sum_{n=m+1}^{c_{j,i}} D_{p_m,q_n}\right) - \sum_{m=1}^{c_{j,i}-1}\sum_{n=m+1}^{c_{j,i}} D_{p_m,q_n} =$$
$$= \left(c_{j,i}^2 - 1\right)\sum_{m=1}^{c_{j,i}} D_{p_j,q_m} - c_{j,i}\sum_{m=1}^{c_{j,i}-1}\sum_{n=m+1}^{c_{j,i}} D_{p_m,q_n}$$

The product:

$$P_{tree_i} \cdot R_{tree_i} = \sum_{j=1}^{n_{tree_i}} \left(NB_j^i + NC_{\{j,k(j)\}}^i\right) \cdot Comb\left(NB_j^i + NC_{\{j,k(j)\}}^i\right) =$$
$$= \sum_{j=1}^{n_{tree_i}} \left(\left(c_{j,i}^2 - 1\right)\sum_{m=1}^{c_{j,i}} D_{p_j,q_m} - c_{j,i}\sum_{m=1}^{c_{j,i}-1}\sum_{n=m+1}^{c_{j,i}} D_{p_m,q_n}\right)$$

And the product:

$$P \cdot R = P_{tree_1} \cdot R_{tree_1} + \ldots + P_{tree_T} \cdot R_{tree_T} =$$
$$= \sum_{i=1}^{T}\sum_{j=1}^{n_{tree_t}} \left(\left(c_{j,i}^2 - 1\right)\sum_{m=1}^{c_{j,i}} D_{p_j,q_m} - c_{j,i}\sum_{m=1}^{c_{j,i}-1}\sum_{n=m+1}^{c_{j,i}} D_{p_m,q_n}\right)$$

$\square$