- Mathematical Models for Systems -

Lecturer: Dr. Sotiris Omiou

**Model definition**

- In order to be able to understand the behavior of systems it is necessary to obtain mathematical models of them.

- A model ship is a scaled down replica of the full-sized ship (there is a constant scaling down of the sizes)

- A photograph can be considered to be a model of the scene that was photographed.

A mathematical model of a system is a “replica” of the relationships between the input(s) and output(s). The actual relationships that exist between the input and output of a system have been replaced by mathematical expressions.
Example:

If there is a linear relationship between the output and input for the motor, then the mathematical model is

\[ \omega = G \cdot V \quad (1) \]

where \( G \) is the constant of proportionality.

The above relationship refers to the **steady-state conditions**.

**Transfer function**

The constant \( G \) is called the *transfer function* or *gain* of the system:

\[
\text{Transfer function } G = \frac{\text{Steady-state output}}{\text{Steady-state input}}
\]

Example: A linear temperature measuring system with an input of 10 °C produces a steady-state output of 5 mV. Such a system has a transfer function

\[ G = 0.5 \text{ mV} / \degree\text{C} \]

The mathematical model of the system is:

**Steady-state output in mV = 0.5 x steady-state input in °C**
Example: Calculate the steady-state output speed for a motor with a transfer function $G = 500 \text{ rev/min}$

Input $\rightarrow$ Motor $\rightarrow$ Output

Voltage $V = 12$ Volts $\rightarrow$ Rotation speed $\omega$

Steady-state output $= G \times$ steady-state input
$= 500 \times 12 = 6000 \text{ rev/min}$

Mathematical models for open-loop systems

Input $\Theta_i$ $\rightarrow$ Element 1 T.F $G_1$ $\rightarrow$ Element 2 T.F $G_2$ $\rightarrow$ Element 3 T.F $G_3$ $\rightarrow$ Output $\Theta_o$

$G_1 = \frac{\Theta_1}{\Theta_i}$  
$G_2 = \frac{\Theta_2}{\Theta_1}$  
$G_3 = \frac{\Theta_o}{\Theta_2}$

The overall transfer function for the system is the output $\Theta_o$ divided by the input $\Theta_i$ or the product of the gains of the elements

$\frac{\Theta_o}{\Theta_i} = \frac{\Theta_1}{\Theta_i} \times \frac{\Theta_2}{\Theta_1} \times \frac{\Theta_o}{\Theta_2}$

Transfer function $= G_1 \times G_2 \times G_3$
Example: The measurement system used with a control system consists of 2 elements with individual transfer functions as in figure. What is the overall transfer function of the measurement system?

Transfer function \( G = 0.1 \times 20 = 2 \text{ mA/Pa} \)

Mathematical models for closed-loop systems (1)

Transfer function \( = \frac{\text{output}}{\text{input}} = \Theta_o / \Theta_i \)
Mathematical models for closed-loop systems (2)

System with Transf. Funct. $G$

Measurement system with Transf. Funct. $H$

Each subsystem within the overall system has its own transfer function:

$$G = \frac{\Theta_0}{e} \quad \quad H = \frac{f}{\Theta_0}$$

Since the error $e$ signal is given by: $e = \Theta_i - f$

we may substitute $e$ and $f$ using the equations above

Mathematical models for closed-loop systems (3)

System with Transf. Funct. $G$

Measurement system with Transf. Funct. $H$

$$\frac{\Theta_0}{G} = \Theta_i - H \cdot \Theta_0 \quad \quad \Theta_0 \left( \frac{1}{G} + H \right) = \Theta_i \quad \quad \Theta_0 \left( \frac{1 + G \cdot H}{G} \right) = \Theta_i$$

Hence the overall transfer function of the closed-loop control system is:

Transfer function $= \frac{\Theta_0}{\Theta_i} = \frac{G}{1 + G \cdot H}$ (2)

* For positive feedback: $1 - G \cdot H$
**Example:** A speed-controlled motor has an amplifier - relay - motor system with a combined transfer function of 600 rev/min per volt and a feedback loop measurement system with a transfer function of 3 mV per rev/min. What is the transfer function of the total system?

The overall transfer function is given by equation (2) as

\[ T. F = \frac{G}{1 + G.H} = \frac{600}{1 + 600 \times 0.003} = 214 \text{ rev/min per volt} \]

**Mathematical models for multi-element closed-loop systems**

Transfer function = \( \frac{\Theta_2}{\Theta_1} = \frac{G_1 \times G_2 \times G_3}{1 + (G_1 \times G_2 \times G_3)H} \) (3)
**Example:** A position-control system used with a machine tool has an amplifier in series with a valve-slider arrangement and a feedback loop with a displacement measurement system. If the transfer functions are as follows, what is the overall T. F. for the control system?

- Amplifier T.F.: 20 mA/V
- Valve-slider T.F.: 12 mm/mA
- Measurement T.F.: 3 V/mm

**Answer:** The amplifier and the valve-slider arrangement are in series so the combined transfer function for the two elements is the product of their separate transfer functions:

\[ \text{T.F for the series elements} = 20 \times 12 = 240 \text{ mm/V} \]

These elements have a feedback loop with a transfer function of 30 mV/mm. Thus the overall transfer function of the control system is

\[ T.F = \frac{G}{1 + G.H} = \frac{240}{1 + 240 \times 0.030} = 29 \text{ mm/V} \]
**Steady-state error**

The steady-state error $E$ of a system is the difference between the output of the system and its input when conditions are steady-state:

$$E = \Theta_o - \Theta_i$$

Since for a system with an overall transfer function of $G_s$

$$G_s = \Theta_o / \Theta_i$$

then

$$E = G_s \Theta_i - \Theta_i = \Theta_i (G_s - 1) \quad (4)$$

For an open-loop system the steady-state error can be written as

$$E = \Theta_i (G_1 G_2 G_3 - 1) \quad (5)$$

where $G_1$, $G_2$, $G_3$ are the transfer functions of individual elements. For the error to be zero then $G_1 G_2 G_3$ must be equal to 1.

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**Steady-state error (cont...)**

For a closed-loop system the steady-state error can be written using equations (2) and (4) as:

$$E = \Theta_i \left( \frac{G}{1 + GH} - 1 \right) \quad (6)$$

where $G$ is the transfer function of the forward path elements ($G=G_1 G_2 G_3$) and $H$ the transfer function of the measurement system.

For the error to be zero we must have $G=1+GH$ so that $G/(1+GH)$ has the value 1.
**Exercise 1:** The figure below shows an open-loop control system with a controller and a motor. The transfer functions of them are also shown in the figure.

a. Calculate the state-state error of the system

b. How will the error change if, as a result of environmental changes, the transfer function of the motor changes by 10%?

**Answer:**

a. Using equation (5), then before any change occur

\[ E = \Theta_i (G_1 G_2 - 1) \]

\[ E = \Theta_i (12 \times 0.10 - 1) = 0.2 \Theta_i \]

b. If there is a 10% change in the transfer function of the motor, to 0.11 rev/min per V, then

\[ E = \Theta_i (12 \times 0.11 - 1) = 0.32 \Theta_i \]

the error has increased by a factor of 16
**Exercise 2:** The figure below shows a closed-loop control system with a controller, a motor and a feedback loop. The transfer functions of them are also shown in the figure.

a. Calculate the state-state error of the system

b. How will the error change if, as a result of environmental changes the transfer function of the motor changes by 10%?

**Answer**

a. Using equation (6), then before any change occurs

\[ E = \Theta_i \left( \frac{G_1 G_2}{1 + G_1 G_2 H} \right)^{-1} \]

\[ E = \Theta_i \left( \frac{12 \times 0.10}{1 + 12 \times 0.10 \times 1.0} \right)^{-1} = -0.45 \Theta_i \]

b. If there is a 10% change in the transfer function of the motor, to 0.11 rev/min per V, then

\[ E = \Theta_i \left( \frac{12 \times 0.11}{1 + 12 \times 0.11 \times 1.0} \right)^{-1} = -0.43 \Theta_i \]

The change in the error is considerably less than the change that occurred with the open-loop system. In this case the system has a much lower sensitivity to environmental changes.
**Exercise 3:** A closed-loop control system has a forward-path transfer function of 10. What should be the transfer function of the feedback path if there is to be zero steady-state error?

**Answer**

a. Using equation (6), then before any change occurs

\[ E = \Theta_i \left( \frac{G}{1+GH} - 1 \right) \]

for \( E \) to be zero then

\[ \frac{G}{1+GH} = 1 \]

hence with \( G = 10 \) then

\[ 10 = 1 + 10H \]

and so \( H \) must be 0.9.